Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion

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1 \Outline of the talk

V_i: vector fields on $\mathbb{R}^n$ $(0 \leq i \leq d)$.

Consider the following RDE driven by a geom. RP x;

$$dy_t = \sum_{i=1}^{d} V_i(y_t)dx_t^i + V_0(y_t)dt$$

with $y_0 = a \in \mathbb{R}^n$.

Here, x is a geom. RP with roughness $p \in (2, 3)$.

$G\Omega_p(\mathbb{R}^d)$ stands for geom. RP space.

(deterministic)

$(w_t)_{0\le t\le 1}$: d-dim frac BM $(\frac{1}{3} < H \le \frac{1}{2})$
$\mathbf{w}$: canonical lift. (Coutin-Qian '02)

$G\Omega_p(\mathbb{R}^d)$-val. r.v. for $p \in (1/H, 3)$.

♠ Set $\mathbf{x} = \mathbf{w}$ in the above RDE. Then, we have something like SDE driven by fBM.

♠ If V_i's satisfy (hypo)ellipticity condition at a, then the law of y_t has a density $p_t(a, a')$ w.r.t. Lebesgue measure da' on $\mathbb{R}^n$.

Aim of the talk: Prove (off-diagonal) short time asymptotic expansion of $p_t(a, a')$ under ellipticity at a + mild assumptions.

Method: T. Lyons' rough path theory + S. Watanabe's distributional Malliavin calculus in (Watanabe '87)

♣ Watanabe's asymptotic theory seems quite powerful even in rough path setting.

♣ When $H = 1/2$, fBM =BM and RDE= SDE of Stratonovich type. Our result recovers most of (Watanabe '87).

Secondary aims of the talk:

• Looking at Watanabe's theory, in particular, his calculations of SDEs, from a viewpoint of RP theory.

• Advertising the power of RP theory, by reproving and extending one of the strongest results in the usual SDE theory.

♣ **When $H = 1/2$, there are so many preceding results.**

Analytic proofs (Note that $p_t(a, a')$ is heat kernel of heat semigroup.) We do not mention this.

Probabilistic proofs via Feynman-Kac formula
In most of them, asymptotics of Laplace-type integrals was computed.

Some famous probabilistic results are;

(0) Molchanov '75 Pinned diffusion process

(1) Bismut Mallaivin calc. of Bismut-type.

(green lecture note '84)

(2) Ben Arous, Léandre, etc...(late 80's-early 90's)

Laplace's method (without Mallaivin calc.)

(3) Watanabe '87 (& '93 with Takanobu)

distributional Mallaivin calc. of Watanabe-type.

(4) Kusuoka-Stroock '91, '94

generalized Mallaivin calc. of KS-type.

Three classes of Gaussian RP. **(A)⊃(B)⊃(C)**

$w = (w^1, \ldots, w^d)$; **conti. Gaussian proc,**
mean 0, **iid-components,** $w_0 = 0$.

$R(s,t) := \mathbb{E}[w_s^1 w_t^1]$; **covariance**

$\mathcal{H}$; **Cameron-Martin space**

(A): **If** R **is of finite 2D** ρ**-variation** $(\exists\rho \in [1,2))$**,**
then w **admits a canonical RP lift with** $p \in [2\rho, 4)$**.**
(Friz-Victoir '10. 2nd/3rd level RP theory.)

(C): fBM with $H \in (1/4, 1/2]$**.**
Essential barrier at $1/4$**. Small barrier at** $1/3$**.**
(Coutin-Qian '02. A prominent example.)

(B) Complementary Young condition:

Gauss proc w in class (A) which satisfies the following;

$\exists p \in [2\rho, 4)$ **and** $\exists q \in [1, 2)$ **s.t.**

(i) $\mathcal{H} \subset C^{q-var}([0,1], \mathbb{R}^d)$

(ii) $1/p + 1/q > 1$ **(Condition for Young integral)**

♣ **This says the two translations (in abstract Wiener space and in geom. RP space) are compatible through RP lifting procedure.**

♣ **(B)⊃(C) by Friz-Victior '06.**

2 Malliavin calculus for RDE

♠ Cass-Friz-(Victoir) '09, '10

A class of Gaussian RP $\ni$ fBM $H \in (1/3, 1/2]$.

- Differentiability in a weak sense i.e., $y_t \in \mathbb{D}_{p,1}^{loc}$
- Malliavin non-degeneracy under Hörmander condition in a weak sense, i.e., $\exists$(Malliavin cov. matrix)$^{-1}$ a.s.

$\Longrightarrow$ $\exists$density of y_t

But, regularity of the density ??.

♠ **Hairer-Pillai '11**

fBM $H \in (1/3, 1/2]$

- **Differentiability i.e., $y_t \in \mathbb{D}_\infty$**
- **Malliavin non-degeneracy under Hörmander condition, i.e.,**
(Malliavin cov. matrix)$^{-1} \in \cap_{1<q<\infty} L^q$

$\Longrightarrow$ **the density $p_t(a, a')$ of y_t is smooth in a'**

Some recent developments

♠ Cass-Hairer-Litterer-Tindel ('12+)

Malliavin non-degeneracy under Hörmander condition for more general Gauss. RPs.

♠ I. ('14)

Differetiability under "complementary Young regularity" condition on Gauss RP

These two results probably enable us to carry out Malliavin calculus on RP space rather smoothly. (Lots of papers will probably be produced)

For fBM with $H \in (1/4, 1/2]$, there already are:

♠ Baudoin-Ouyang-Zhang '13+

Varadhan's estimate i.e., short time asymptotics of $\log p_t(a, a')$

♠ Baudoin-Ouyang-Zhang '13+

Smoothing effect of "heat semigroup" under Kusuoka father's UFG condition.

♠ Baudoin-E. Nualart-Ouyang-Tindel '14+

Positivity of the density $p_t(a, a')$.

♠ I. '14. (This work. $H > 1/3$)

short time off-diagonal asymptotics of $p_t(a, a')$

3 Index sets

Index sets for the asymptotics are quite complicated.
Set

$$\begin{aligned}\boldsymbol{\Lambda_1} &= \left\{\boldsymbol{n_1 + \frac{n_2}{H} \mid n_1, n_2 \in \mathbb{N}}\right\}, \\ &= \left\{\mathbf{0,\ 1,\ 2,\ \frac{1}{H},\ 3,\ 1 + \frac{1}{H},\ 4, \ldots}\right\}\end{aligned}$$

No wonder why Λ_1 appears, because we consider the scaled RDE as always: For $0 < \varepsilon \leq 1$,

$$dy_t^\varepsilon = \sum_{i=1}^{d} V_i(y_t^\varepsilon)\varepsilon dw_t^i + V_0(y_t^\varepsilon)\varepsilon^{1/H} dt$$

The laws of $(y_t^\varepsilon)_{0\leq t\leq 1}$ and $(y_{\varepsilon^{1/H}t})_{0\leq t\leq 1}$ are the same, due to the scale invariance of fBM.

We also set

$$\boldsymbol{\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\}}$$
$$\boldsymbol{= \left\{0,\ 1,\ \frac{1}{H} - 1,\ 2,\ \frac{1}{H},\ 3, \ldots\right\}}$$

and

$$\boldsymbol{\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1\}\}}$$
$$\boldsymbol{= \left\{0,\ \frac{1}{H} - 2,\ 1,\ \frac{1}{H} - 1,\ 2, \ldots\right\}.}$$

Next, we set

$$\begin{aligned}\boldsymbol{\Lambda_3} &:= \mathbb{N}\langle \boldsymbol{\Lambda_2} \rangle \\ &= \{\boldsymbol{a_1 + a_2 + \cdots + a_m} \mid \\ &\qquad \boldsymbol{m} \in \mathbb{N}_+ \textbf{ and } \boldsymbol{a_1, \ldots, a_m} \in \boldsymbol{\Lambda_2}\}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Lambda_3'} &:= \mathbb{N}\langle \boldsymbol{\Lambda_2'} \rangle \\ &= \{\boldsymbol{b_1 + b_2 + \cdots + b_m} \mid \\ &\qquad \boldsymbol{m} \in \mathbb{N}_+ \textbf{ and } \boldsymbol{b_1, \ldots, b_m} \in \boldsymbol{\Lambda_2'}\}\end{aligned}$$

Finally, we set

$$\Lambda_4 := \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}$$
$$=: \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$$

This index set Λ_4 appears in our main theorem i.e., the off-diagonal asymptotic expansion.

If $H = 1/2$ or the drift $\equiv 0$, all these index sets above are just $\mathbb{N}$.

4 Assumptions

(A1): The set of vectors $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbb{R}^n$.

(Elipticity at the starting point a)

$\Longrightarrow$ $\exists$ denisty $p_t(a, a')$ for any $t > 0$.

i.e., $\mathbb{P}(y(t, a) \in U) = \int_U p_t(a, a') da'$.

$\mathcal{H} = \mathcal{H}^H$: Cameron-Martin space of fBm (w_t).

$\Longrightarrow \forall \gamma \in \mathcal{H}$ is of finite q-variation for

$$q = (H + 1/2)^{-1} \in [1, 2).$$

For $\gamma \in \mathcal{H}$, we denote by $\phi_t^0 = \phi_t^0(\gamma)$ be the solution of the following Young ODE;

$$d\phi_t^0 = \sum_{i=1}^{d} V_i(\phi_t^0) d\gamma_t^i \quad \text{with} \quad \phi_0^0 = a \in \mathbb{R}^n.$$

For $a' \neq a$, set $K_a^{a'} := \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}$.

(A2): $\exists!\,\bar{\gamma} \in K_a^{a'}$ **which minimizes $\mathcal{H}$-norm.**

Note that, if "inf" exists, then
$\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$.

We also assume that $\|\cdot\|_{\mathcal{H}}^2/2$ is not so degenerate at this $\bar{\gamma}$ in the following sense.

(A3): At $\bar{\gamma}$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathcal{H}}^2/2$ is strictly positive, that is, if $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto \gamma(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $\gamma(0) = \bar{\gamma}$ and $\gamma'(0) \neq 0$, then $(d/du)^2|_{u=0}\|\gamma(u)\|_{\mathcal{H}}^2/2 > 0$.

♣ Under (A1)–(A2), Assumption (A3) is equivalent to exponential integrability of certain quadratic Wiener functional, which appears in the proof of off-diagonal asymptotics.

Remark: Assume (A1) at a. If a' is sufficiently close to a, then (A2)–(A3) hold.

($\because$ The implicit function theorem $\times 2$.)

5 Main theorem

Assume V_i are of C_b^∞. Consider the RDE

$$dy_t = \sum_{i=1}^{d} V_i(y_t) d\mathrm{w}_t^i + V_0(y_t) dt$$

with $y_0 = a \in \mathbb{R}^n$.
Here, $\mathbf{w}$ stands for fractional Brownian RP with Hurst parameter $1/3 < H \leq 1/2$.

♠ Our main theorem is a "rough path version" of Watanabe '87 (and basically parallel to it.)

[Theorem] Assume $a \neq a'$ and **(A1)–(A3)**.
Then, as $t \searrow 0$, we have;

$$p(t,a,a') \sim \exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2t^{2H}}\Big)\frac{1}{t^{nH}} \times \{\alpha_0 + \alpha_{\lambda_1} t^{\lambda_1 H} + \alpha_{\lambda_2} t^{\lambda_2 H} + \cdots\}$$

for certain real constants α_{λ_j} $(j = 0, 1, 2, \ldots)$.
Here, $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ are all the elements of Λ_4 in increasing order.

Remark:

♠ When $H = 1/2$ or $V_0 \equiv 0$, then $\Lambda_4 = \mathbb{N}$.
But, in fact, the odd terms cancel out.
So, the index set is actually $2\mathbb{N}$.

♣ When $H = 1/2$, our argument can be regarded as a rough path proof of Watanabe ('87).
Compared to it, the large deviation part (i.e., the localization procedure) looks quite straight forward.

(The following two cases are not covered, yet.
But, we believe those are not extremely important.)

(a): In this paper the ellipticity assumption (A1) is assumed. In Watanabe '87, something like "step 2-hypoellipticity" case was also studied.
(We simply did not try this case.)

(b): The condition on vector fields in Watanabe '87 is not C_b^∞, but is as follows: "For all $k = 1, 2, \ldots$ and $0 \le i \le d$, $\|\nabla^k V_i\|$ is bounded."
(i.e., V_i itself may have linear growth.)

(cf. I. Bailleul recently solved RDEs with such coefficients.)

6 Outline of Proof

The scaled RDE; for $\varepsilon \in (0, 1]$,

$$dy_t^\varepsilon = \sum_{i=1}^{d} V_i(y_t^\varepsilon)\varepsilon dw_t^i + V_0(y_t^\varepsilon)\varepsilon^{1/H} dt$$

$y_1^\varepsilon \approx y_{\varepsilon^{1/H}}$ in law, $\quad$ ($\because$ scale invariance of fBM).
Its CM shift by $\bar{\gamma} \in \mathcal{H}$ in (A2);

$$d\tilde{y}_t^\varepsilon = \sum_{i=1}^{d} V_i(\tilde{y}_t^\varepsilon) d(\varepsilon w_t^i + \bar{\gamma}) + V_0(\tilde{y}_t^\varepsilon)\varepsilon^{1/H} dt$$

♠ Taylor expansion of Lyons-Itô map around $\bar{\gamma}$

As $\varepsilon \searrow 0$, there exist $\phi^0(\bar{\gamma}), \phi^{\kappa_i}(\mathrm{w}, \bar{\gamma})$ s.t.

$$\begin{aligned}\tilde{y}_1^\varepsilon &\sim \phi_1^0 + \varepsilon^{\kappa_1}\phi_1^{\kappa_1} + \varepsilon^{\kappa_2}\phi_1^{\kappa_2} + \cdots \\ &= \phi_1^0 + \varepsilon^1\phi_1^1 + \varepsilon^2\phi_1^2 + \varepsilon^{1/H}\phi_1^{1/H} + \cdots,\end{aligned}$$

where the index set is $\Lambda_1 = \mathbb{N} + (1/H)\mathbb{N}$.
(Both in deterministic sense and $\mathbb{D}_\infty$ sense.)

Set $R^{1,\varepsilon} := \tilde{y}_1^\varepsilon - \phi_1^0$ and $R^{2,\varepsilon} := \tilde{y}_1^\varepsilon - \phi_1^0 - \varepsilon^1\phi_1^1$
The index sets for $R^{1,\varepsilon}/\varepsilon$ and $R^{2,\varepsilon}/\varepsilon^2$
are Λ_2 and Λ_2', respectively.

- The deterministic sense was shown in I.-Kawabi ('07) or I. ('10).

- We have to prove $\mathbb{D}_\infty$ sense.

Sufficient to prove L^q sense for $1 < \forall q < \infty$.

Moment estimate of $R^{1,\varepsilon} = \tilde{y}^\varepsilon - \phi^0(\bar{\gamma})$, i.e., the first step of the induction, is most difficult.

Thanks to an integrability lemma by Cass-Litterer-Lyons ('13), we can prove that part.

♠ Watanebe's pullback of δ-function

$$p(\varepsilon^{1/H}, a, a') = \mathbb{E}\big[\delta_{a'}(y_{\varepsilon^{1/H}})\big] = \mathbb{E}\big[\delta_{a'}(y_1^{\varepsilon})\big]$$

$$= \mathbb{E}\big[\delta_{a'}(y_1^{\varepsilon})\chi_\eta(\varepsilon, w)\big] + \text{(a small term).}$$

Here, $\chi_\eta(\varepsilon, w)$ is a $\mathbb{D}_\infty$-functional which looks like the indicator of a small ball of a certain radius $\eta > 0$ centered at $\bar{\gamma}$ (on RP space with Besov norms.)

By Schilder-type large deviation on RP space (= the domain of Lyons-Itô map), the second term above is negligible.

By CM formula, the first term equals to

$$\exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\Big)\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}\langle\bar{\gamma}, w\rangle\Big)\delta_{a'}(\tilde{y}_1^{\varepsilon})\chi_\eta\Big(\varepsilon, w+\frac{\bar{\gamma}}{\varepsilon}\Big)\Big]$$

Here, $w = \mathrm{w}^1$, the original fBM.
But, $\chi_\eta(\varepsilon, w + \bar{\gamma}/\varepsilon)$ does not contribute to the asymptotic expansion since it is of the form $1 + O(\varepsilon^N)$ for any large $N \in \mathbb{N}$.

♣ So, it suffices to expand

$$\exp\Big(-\frac{1}{\varepsilon}\langle\bar{\gamma}, w\rangle\Big) \quad \text{and} \quad \delta_{a'}(\tilde{y}_1^{\varepsilon})$$

By Lagrange multiplier method, $\exists \boldsymbol{\nu} \in \mathbb{R}^n$ s.t.

$$\langle \bar{\gamma}, \mathrm{w} \rangle = \langle \boldsymbol{\nu}, \phi_1^1(\bar{\gamma}, \mathrm{w}) \rangle_{\mathbb{R}^n}$$

Under the condition $\tilde{y}_1^{\varepsilon} = a'$ $(= \phi_1^0(\bar{\gamma}))$, we have $\phi_1^1 + \varepsilon^{-1} R_1^{2,\varepsilon} = 0$ and hence

$$\exp\Big(-\frac{1}{\varepsilon}\langle \bar{\gamma}, w \rangle\Big) = \exp\Big(\frac{\langle \boldsymbol{\nu}, R_1^{2,\varepsilon} \rangle}{\varepsilon^2}\Big).$$

Index set for $R_1^{2,\varepsilon}/\varepsilon^2$ is $\boldsymbol{\Lambda}_2'$,
$\Longrightarrow$ Index set for RHS is $\boldsymbol{\Lambda}_3'$

[Remark] **(i)** **This expansion takes place in** $\tilde{\mathbb{D}}_\infty := \cap_{k=0}^{\infty} \cup_{1<q<\infty} \mathbb{D}_{q,k}$ **topology.**

(ii) **Loosely speaking,** $\exp\big(\langle \bar{\nu}, R_1^{2,\varepsilon}\rangle / \varepsilon^2\big)$ **a quadratic Wiener functional on** "exp".

Integrabiity is quite subtle, even we have Fernique's theorem.

(A3) is assumed for this kind of quantity to be integrable.

Since $\phi_1^0(\bar{\gamma}) = a'$, we have

$$\delta_{a'}(\tilde{y}_1^\varepsilon) = \delta_0\Big(\varepsilon \cdot \frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\Big) = \varepsilon^{-n}\delta_0\Big(\frac{R_1^{1,\varepsilon}}{\varepsilon}\Big).$$

- Index set for the expansion of $R_1^{1,\varepsilon}/\varepsilon$ is Λ_2.
- $R_1^{1,\varepsilon}/\varepsilon$ is uniformly non-degenerate in the sense of Malliavin under (A1).
- So, RHS admits asymptotic expansion in the space of Watanabe disributions whose index set is Λ_3. (Watanabe's asymtotic theorem)

$$
\begin{aligned}
&p(\varepsilon^{1/H}, a, a') \\
&\sim \exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\Big)\mathbb{E}\Big[\exp\Big(-\frac{\langle\bar{\gamma}, w\rangle}{\varepsilon}\Big)\delta_{a'}(\tilde{y}_1^{\varepsilon})\chi_{\eta}(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon})\Big] \\
&\sim \exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\Big) \\
&\qquad \times \frac{1}{\varepsilon^n}\{\alpha_0 + \alpha_{\lambda_1}\varepsilon^{\lambda_1} + \alpha_{\lambda_2}\varepsilon^{\lambda_2} + \cdots\}
\end{aligned}
$$

Here, $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ are all the elements of $\Lambda_4 := \Lambda_3 + \Lambda_3'$ in increasing order.