Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion

Yuzuru Inahama (Nagoya University)

Abstract: We study a stochastic differential equation in the sense of rough path theory driven by fractional Brownian rough path with Hurst parameter H $(1/3 < H \leq 1/2)$ under the ellipticity assumption at the starting point. In such a case, the law of the solution at a fixed time has a kernel, i.e., a density function with respect to Lebesgue measure. (See [1]). In this paper we prove a short time *off-diagonal* asymptotic expansion of the kernel under mild additional assumptions. Our main tool is Watanabe's distributional Malliavin calculus developped in [2]. Unlike some other works on asymptotics for SDEs driven by fBm, our RDE (1) has a drift term. This makes the asymptotic expansion quite comlicated. Note also that when $H = 1/2$, SDE (1) is just a Stratonovich SDE driven by the usual Brownian motion. Therefore, our result can be regards as a generaliztion of Watanabe [2].

Let $(w_t)_{t\geq 0} = (w_t^1, \ldots, w_t^d)_{t\geq 0}$ be the d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (1/3, 1/2]$. Let $V_i : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞, that is, V_i is a bounded smooth function with bounded derivatives of all order $(0 \leq i \leq d)$. We consider the following (random) rough differential equation (RDE) driven by fractional Brownian rough path, i.e., the natural lift of fBm (w_t);

$$dy_t = \sum_{i=1}^d V_i(y_t)dw_t^i + V_0(y_t)dt \qquad \text{with} \qquad y_0 = a \in \mathbf{R}^n. \tag{1}$$

We will sometimes write $y_t = y_t(a)$ etc. to make explicit the dependence on a.

First, we assume the ellipticity of the coefficient of (1) at the starting point $a \in \mathbf{R}^n$.

(A1): The set of vectors $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbf{R}^n$.

Under Assumption **(A1)**, the law of the solution y_t has a density $p_t(a, a')$ with respect to the Lebesgue measure da' on $\mathbf{R}^n$ for any $t > 0$. Let $\mathcal{H} = \mathcal{H}^H$ be the Cameron-Martin space of fBm (w_t). For $\gamma \in \mathcal{H}$, we denote by $\phi_t^0 = \phi_t^0(\gamma)$ be the solution of the following Young ODE;

$$d\phi_t^0 = \sum_{i=1}^d V_i(\phi_t^0)d\gamma_t^i \qquad \text{with} \qquad \phi_0^0 = a \in \mathbf{R}^n.$$

Set, for $a \neq a'$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

If we assume **(A1)** for all a, this set $K_a^{a'}$ is not empty. If $K_a^{a'}$ is not empty, it is a Hilbert submanifold of $\mathcal{H}$. It is known that $\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$. Now we introduce the following assumption;

(A2): $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathcal{H}$-norm exists uniquely.

In the sequel, $\bar{\gamma}$ denotes the minimizer in Assumption **(A2)**. We also assume that the Hessian of $\|\cdot\|_{\mathcal{H}}^2/2$ is not so degenerate at $\bar{\gamma}$ in the following sense.

(A3): At $\bar{\gamma}$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathcal{H}}^2/2$ is strictly larger than $\mathrm{Id}_{\mathcal{H}^H}/2$ in the form sense. More precisely, If $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$, then $(d/du)^2|_{u=0}\|f(u)\|_{\mathcal{H}}^2/2 > 0$.

Now,we introduce several index sets for the exponent of the small parameter $\varepsilon := t^H > 0$, which will be used in the asymptotic expansion. Unlike in many preceding papers, index sets in this paper are not (a constant multiple of) $\mathbf{N} = \{0, 1, 2, \dots\}$ and are quite complicated.

Set $\Lambda_1 = \{n_1 + \frac{n_2}{H} \mid n_1, n_2 \in \mathbf{N}\}$. We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ all the elements of Λ_1 in increasing order. Several smallest elements are explicitly given as follows; $\kappa_1 = 1, \quad \kappa_2 = 2, \quad \kappa_3 = \frac{1}{H}, \quad \kappa_4 = 3, \quad \kappa_5 = 1 + \frac{1}{H}, \dots$ As usual, using the scale invariance (i.e., self-similarity) of fBm, we will study the scaled version of (1). From its explicit form, one can easily see why Λ_1 appears.

We also set $\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\} = \{0, 1, \frac{1}{H} - 1, 2\, \frac{1}{H}, 3 \dots\}$ and $\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1\}\} = \{0, \frac{1}{H} - 2, 1, \frac{1}{H} - 1, 2, \dots\}$. Next we set

$$\Lambda_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda_2\}.$$

In the sequel, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ stands for all the elements of Λ_3 in increasing order. Similarly,

$$\Lambda_3' = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda_2'\}.$$

In the sequel, $\{0 = \rho_0 < \rho_1 < \rho_2 < \cdots\}$ stands for all the elements of Λ_3' in increasing order. Finally, $\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}$. We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ all the elements of Λ_4 in increasing order.

Below we state two main results of ours, which are basically analogous to the corresponding ones in Watanabe [2]. However, there are some differences. First, the exponents on the shoulder of t are not (a constant multiple of) natural numbers. Second, cancellation of "odd terms" as in p. 20 and p. 34, [2] does not happen in general in our case. (If the drift term in RDE (1) is zero or if $H = 1/2$, then this kind of cancellation takes place).

The following is a short time asymptotic expansion of the diagonal of the kernel function. This is much easier than the off-diagonal case.

Theorem 1 *Assume* **(A1)**. *Then, the diagonal of the kernel $p(t, a, a)$ admits the following asymptotics as $t \searrow 0$;*

$$p(t, a, a) \sim \frac{1}{t^{nH}} \left(c_0 + c_{\nu_1} t^{\nu_1 H} + c_{\nu_2} t^{\nu_2 H} + \cdots \right)$$

for certain real constants c_{ν_j} $(j = 0, 1, 2, \dots)$. Here, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ are all the elements of Λ_3 in increasing order.

We also have off-diagonal short time asymptotics of the kernel function. This is our main result.

Theorem 2 *Assume* **(A1)**–**(A3)**. *Then, we have the following asymptotic expansion as $t \searrow 0$;*

$$p(t, a, a') \sim \exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2t^{2H}} \Big) \frac{1}{t^{nH}} \{ c_{\lambda_0} + c_{\lambda_1} t^{\lambda_1 H} + c_{\lambda_2} t^{\lambda_2 H} + \cdots \}$$

for certain real constants c_{λ_j} $(j = 0, 1, 2, \dots)$. Here, $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ are all the elements of Λ_4 in increasing order.

References

[1] Hairer, M.; Pillai, N.; Ann. Probab. 41 (2013), no. 4, 2544–2598.

[2] Watanabe, S.; Ann. Probab. 15 (1987), no. 1, 1–39.