A PROOF OF L^p–LOGARITHMIC SOBOLEV INEQUALITY VIA SEVERAL APPROXIMATIONS

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This talk is based on [7].
For a smooth enough function $f \geq 0$ on $\mathbb{R}^n$, we define the entropy of f with respect to the Lebesgue measure by

$$\mathrm{Ent}(f) = \int f(x) \log f(x) dx - \int f(x) dx \ \log \int f(x) dx.$$

In this talk, the integral without its domain is always understood as the one over $\mathbb{R}^n$, and we interpret that $0 \log 0 = 0$.

Let $p \geq 1$. We denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions f on $\mathbb{R}^n$ such that f and $|Df|$ (the Euclidean length of the gradient Df of f) are in $L^p(\mathbb{R}^n)$. For $f \in W^{1,p}(\mathbb{R}^n)$, the following L^p–logarithmic Sobolev inequality was shown for $p = 2$ by [11], $p = 1$ by [10], and $1 < p < n$ by [6]:

$$\mathrm{Ent}(|f|^p) \leq \frac{n}{p} \int |f(x)|^p dx \ \log \left(L_p \frac{\displaystyle\int |Df(x)|^p \, dx}{\displaystyle\int |f(x)|^p \, dx} \right). \tag{1}$$

Here,

$$L_p = \begin{cases} \dfrac{p}{n} \left(\dfrac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left(\dfrac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n \frac{p-1}{p} + 1\right)} \right)^{p/n}, & p > 1, \\ \dfrac{1}{n} \pi^{-1/2} \left[\Gamma\left(\dfrac{n}{2} + 1 \right) \right]^{1/n}, & p = 1. \end{cases} \tag{2}$$

This is the best possible constant satisfying (1) for $1 \leq p < n$ (cf. [1, 6]).

For a general $p > 1$, with a deep insight, Gentil [9, Theorem 1.1] gave inequality (1) by using a hypercontractivity inequality for the unique viscosity solution to the Cauchy problem of a Hamilton-Jacobi equation. However, his proof for inequality (1) is valid for a special class of functions f in $W^{1,p}(\mathbb{R}^n)$.

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Our aim of this talk is to bridge the gap in the proof of [9, Theorem 1.1] and provide a supplementary proof of inequality (1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. The strategy of our proof is the following:

First, we show (1) for $f \in W^{1,p}(\mathbb{R}^n)$ such that

$$f \in C^1(\mathbb{R}^n),\ 0 < f \leq 1 \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n. \tag{3}$$

Second, we approximate $f \in W^{1,p}(\mathbb{R}^n)$ by a sequence of functions satisfying (3) by several steps. This is the key point to derive (1). An important tool is the following Fatou–type inequality: if a family $\{f_\epsilon\}_{0<\epsilon<1}$ of nonnegative and measurable functions on $\mathbb{R}^n$ approximates a function f in some sense, then

$$\liminf_{\epsilon \to 0+} \int f_\epsilon(x)^p \log f_\epsilon(x) dx \geq \int f(x)^p \log f(x) dx. \tag{4}$$

Finally, by using these approximations, we show that L^p–logarithmic Sobolev inequality (1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$.

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