An Integration by Parts on "Space of Loops"

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Outline

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Introduction.

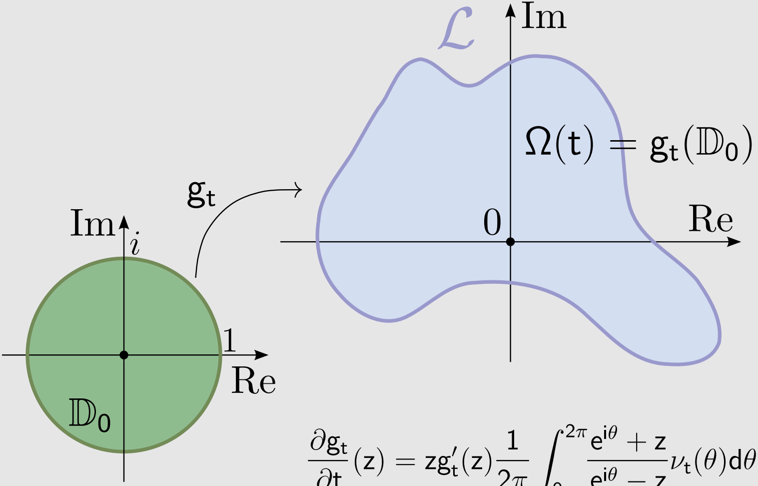
The Loewner-Kufarev equation. I

For a given increasing/decreasing subordination chain $(\Omega(t))_{0\le t\le T}$ in the complex plane $\mathbb{C}$, each of which includes 0 as an interior point, it is known that there exists a family $(\nu_t(\theta)d\theta)_{0\le t\le T}$ of measure on the unit circle S^1 such that

$$\frac{\partial g_t}{\partial t}(z) = z g_t'(z) \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\nu_t(\theta)\mathrm{d}\theta}_{=:p(t,z)}$$

and $\mathrm{Re}\, p(t,z)$ is positive/negative if $\Omega(t)$ is increasing/decreasing, where g_t : the unit disk $\mathbb{D}_0 \to \Omega(t)$ is the unique conformal mapping with $g_t(0)=0$ and $g_t'(0)>0$ for a.a. $t\in[0,T]$ (Pommerenke).

The Loewner-Kufarev equation. II



$$\frac{\partial g_t}{\partial t}(z) = z g_t'(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(\theta) d\theta$$

By extending g_t continuously, we have a family of loops

$$g_t : S^1 = \partial\mathbb{D}_0 \to \partial\Omega(t), \quad 0 \le t \le T.$$

Therefore, randomizing the Loewner-Kufarev equation, we would have a probability measure on loop space

$$\big\{\text{(image of) continuous } S^1 \to \mathbb{C}^*\big\}$$

(-valued path space).

In fact, we consider a **stochastic Loewner-Kufarev equation** in subsequent sections, and then we study its properties.

Background. I

Several authors have investigated constructions and properties of measures on loops, and conjectured existence of such measures which admit some prerequired properties. These researches seem to be initiated by Malliavin(s).

- Malliavin (1999)
  - — constructed a canonic diffusion "on" $\mathrm{Diff}_+(S^1)$. Solving a corresponding conformal welding problem, this would induce a probability measure on loops through the (extended) Kirillov's bijection.
- Shavglidze (2000)
  - — constructed a measure on C^1-diffeos. $\mathrm{Diff}^1_+(S^1)$. This measure is probed to be quasi-invariant under $\mathrm{Diff}_+(S^1) \curvearrowright \mathrm{Diff}^1_+(S^1)$ and then the Schwarzian derivative appears in the density, which is striking the fact that the Schwarzian derivative is a 1-cocycle on $\mathrm{Diff}_+(S^1)$.

Background. II

Then our main result is

Theorem

Malliavin's canonic diffusion has a similar defining equation to a stochastic Loewner-Kufarev equation.

Background. III

- Airault-Malliavin (2001)
  — began to seek a Gaussian realization of L^2-reps of Virasoro algebra. This was inspired by Segal (1963), Bargmann (1961) and Frenkel (1984), in which they gave Gaussian realizations of L^2-reps of ∞-dim. groups/algebras.

It has been known by

- Kirillov-Yuriev (1988)
  — that $\exists$ a representation of Virasoro algebra on univalent functions on $\mathbb{D}_0$ (rather than $\mathbb{D}_0 \setminus \{0\}$ and strictly speaking, on the "coefficient body" of univalent functions).

However,

- Airault-Malliavin-Thalmaier (2002)
  — $\exists$ **no** **probability** measures on univalent functions, which make Kirillov-Yuriev's one be a unitary rep. of Virasoro algebra.

Background. IV

Inspired by Malliavin's work and SLE theory,

- Kontsevich-Suhov (2007)
  - — conjectured the unique existence of non-zero assignment

$$\begin{array}{c}\text{Riemann}\\ \text{surface } \Sigma\end{array} \mapsto \begin{array}{c}(|\mathrm{Det}_\Sigma|)^{\otimes c}\text{-valued}\\ \text{measure } \lambda_\Sigma,\end{array}$$

which is locally conformally covariant with parameter $c \in (-\infty, 1]$. Furthermore, they proposed a reduction of this problem, to construct a scalar measure on simple loops in $\mathbb{C}^*$, surrounding the origin, satisfying a restriction covariance property. They mentioned that this property has an infinitesimal version, infinitesimal restriction covariance property, formulated by a family of **integration by parts formulae** on space of loops. At this stage, in belief, several natures of Virasoro algebra or Virasoro-Bott group should appear, which is the main motivation of our study.

Background. V

- Werner (2008)
  — constructed such an assignment when $c = 0$. However, in the case $c = 0$, Kontsevich-Suhov's framework reduces to be trivial (everything is conformally invariant rather than covariant), which implies that one would not be able to see any Virasoro nature.
- Benoist-Dubédat
  — announced that they solved the existence part of the conjecture when $c = -2$.

Background. VI

One of our results can be stated very roughly as follows.

Theorem (rough version.)

A stochastic Loewner-Kufarev equation induces a probability measure $\mathbf{P}$ *on "space of loops" such that*

$$\text{“}\int_{\text{Loops}}(L_2F)(\mathcal{L})\,\mathbf{P}(\mathrm{d}\mathcal{L}) = \int_{\text{Loops}} F(\mathcal{L})\mathcal{S}_\phi(0)\,\mathbf{P}(\mathrm{d}\mathcal{L})\,\text{”},$$

for any "polynomial function" F*, where*

- $\triangleright$ $(L_2F)(\mathcal{L}) = \dfrac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} F\Big(\exp(-uz^3\dfrac{\mathrm{d}}{\mathrm{d}z})(\mathcal{L})\Big)$,
- $\triangleright$ ϕ *is the unique conformal mapping from* $\mathbb{D}_0$ *to the domain surrounded by* $\mathcal{L}$ *and including the origin with* $\phi(0) = 0$ *and* $\phi'(0) > 0$,
- $\triangleright$ $\mathcal{S}_\phi(z) := \frac{\phi'''(z)}{\phi''(z)} - \frac{3}{2}\frac{\phi''(z)^2}{\phi'(z)^2}$ *: the Schwarzian derivative of* ϕ.

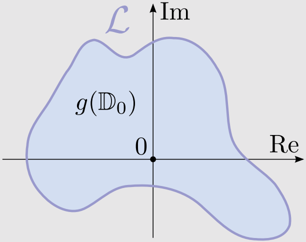
Background. VII

The practical IbP will be established on the state space $\mathbb{R} \times \mathbb{C}^{\mathbb{N}}$ rather than the genuine loop space. However, the spirit is on the space of loops through the following embeddings:

Space of Loops $\hookrightarrow$ Space of

$$g(z) = C(z + c_1 z^2 + c_2 z^3 + \cdots) \quad \text{with} \begin{cases} g(0) = 0, \\ g'(0) > 0 \end{cases}$$

$\rotatebox{90}{$\in$}$



$$\Big\downarrow$$

$$\mathbb{R} \times \mathbb{C}^{\mathbb{N}}$$

$\rotatebox{90}{$\in$}$

$$(C, c_1, c_2, \cdots)$$

Introducing a Stochastic Loewner-Kufarev Equation.

The Loewner-Kufarev equation, revisited. I

Recall that the Loewner-Kufarev equation forms as

$$\frac{\partial g_t}{\partial t}(z) = z g_t'(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(\theta) \mathrm{d}\theta.$$

- The input $(\nu_t)_{0 \le t \le T}$ has a meaning of "density" of the boundary variation $(\partial g_t(\mathbb{D}_0))_{0 \le t \le T}$,

and then, following Friedrich's idea, we rewrite the equation as

$$\dot{g}_t = -\mathcal{L}_{\nu_t}(g_t)$$

where the "vector field" $\mathcal{L}_{\nu_t}$ acts on univalent functions by

$$(\mathcal{L}_{\nu_t} f)(z) := -z f'(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(\theta) \mathrm{d}\theta.$$

The Loewner-Kufarev equation, revisited. II

Assuming a sufficient regularity on $\nu_t(\theta)$, we write its Fourier series as

$$\nu_t(\theta) = a_0(t) + \sum_{k=1}^{\infty} \left\{ a_k(t) \cos(k\theta) + b_k(t) \sin(k\theta) \right\}.$$

Then the Loewner-Kufarev equation can be written as

$$\dot{g}_t = -a_0(t)\mathcal{L}_{\mathbf{1}}(g_t) - \sum_{k=1}^{\infty} \left\{ a_k(t)\mathcal{L}_{\cos(k\cdot)}(g_t) + b_k(t)\mathcal{L}_{\sin(k\cdot)}(g_t) \right\},$$

where $\mathbf{1}(\theta) \equiv 1$. Although the Loewner-Kufarev dynamics is not a linear system, we can heuristically understand from the last equation that fundamental loops (or closed strings), each of which is the image of S^1 mapped by the solution of the system associated to each of $\mathcal{L}_{\mathbf{1}}$, $\mathcal{L}_{\cos(k\cdot)}$ and $\mathcal{L}_{\sin(k\cdot)}$, are piled with weights a_k's and b_k's at the infinitesimal level, and accordingly give a loop $g_t(S^1)$.

Randomizing the Loewner-Kufarev equation. I

This observation turns to make us give randomness to the weights a_k and b_k to construct a measure on the space of loops. We experimentally choose a singular $\nu_t(\theta)$ (recall that $-\nu_t(\theta) = \delta_{e^{-iB_t}}(\theta)$ in the radial SLE case) as $a_0(t) = \alpha_0^{-1} t$ and

$$a_k(t) = \text{``}\alpha_k^{-1} \dot{B}_t^{(k,1)}\text{''} \quad b_k(t) = \text{``}-\alpha_k^{-1} \dot{B}_t^{(k,2)}\text{''}$$

for $k \geq 1$, where

- $(B_t^{(k,1)}, B_t^{(k,2)})$, $k \geq 1$: indep. 2-dim. BMs,
- $(\alpha_k)_{k \geq 1}$: positive real numbers.

Randomizing the Loewner-Kufarev equation. II

Using the relations, for $k \geq 1$ and $|z| < 1$,

- $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\cos(k\theta)\mathrm{d}\theta = z^k$,
- $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\sin(k\theta)\mathrm{d}\theta = -iz^k$,

our Loewner-Kufarev equation becomes

$$\text{“}\frac{\partial g_t}{\partial t}(z) = zg_t'(z)\Big\{\alpha_0^{-1}t + \sum_{k=1}^{\infty}\alpha_k^{-1}(\dot{B}_t^{(k,1)} + i\dot{B}_t^{(k,2)})z^k\Big\}\text{”}.$$

This motivates us to consider the following SDE

$$\mathrm{d}g_t(z) = zg_t'(z)\{\mathrm{d}X_t^0 + \sum_{k=1}^{\infty} z^k \mathrm{d}X_t^k\}, \quad g_0(z) \equiv z \in \mathbb{D}_0,$$

where $X_t^0 = \alpha_0^{-1}t$, $X_t^k = \alpha_k^{-1}Z_t^k$ for $k \geq 1$ and $Z_t^1, Z_t^2, \cdots$ are infinitely many indep. complex BMs.

Randomizing the Loewner-Kufarev equation. III

Remark.

(i) Our SDE

$$\mathrm{d}g_t(z) = zg_t'(z)\big\{\mathrm{d}X_t^0 + \sum_{k=1}^{\infty} z^k \mathrm{d}X_t^k\big\}, \quad g_0(z) \equiv z \in \mathbb{D}_0,$$

is not usual because its diffusion- and drift- coefficients involve the derivative of the stochastic flow. Therefore the classical arguments to deal with typical SDEs can not be applied.

(ii) The parameters α_k's should be taken suitably so that the RHS converges.

A Comparison with Malliavin's Canonic Diffusion.

Malliavin's canonic diffusion. I

Malliavin (1999) constructed a diffusion "on" $\mathrm{Diff}_+(S^1)$ by considering the following SDE

$$\mathrm{d}\sigma_t = \sum_{k=1}^{\infty} \frac{c_k(\sigma_t)\circ \mathrm{d}x_t^{(k,1)} + s_k(\sigma_t)\circ \mathrm{d}x_t^{(k,2)}}{\sqrt{hk + \frac{c}{12}(k^3 - k)}}$$

on $\mathrm{Diff}_+(S^1)$, where $h > 0$ and $c > 0$. The vector fields $\{c_k, s_k\}_{k=1}^{\infty}$ on $\mathrm{Diff}_+(S^1)$ are given by

$$c_k(\sigma) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{e}^{\varepsilon\cos(k\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}} \circ \sigma, \quad s_k(\sigma) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{e}^{\varepsilon\sin(k\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}} \circ \sigma$$

for $\sigma \in \mathrm{Diff}_+(S^1)$, and $x_t^k = (x_t^{(k,1)}, x_t^{(k,2)})$, $k \geq 1$ are indep. 2-dim. BMs.

Malliavin's canonic diffusion. II

Once we specify an initial diffeomorphism $\sigma \in \mathrm{Diff}_+(S^1)$, one has a diffusion $(\sigma_t)_{t \geq 0}$ on **homeomorphism group** of S^1 rather than $\mathrm{Diff}_+(S^1)$. This diffusion is called the **canonic diffusion "on" $\mathbf{Diff_+(S^1)}$**.

Comparison with Malliavin's canonic diffusion. I

Theorem

Malliavin's canonic diffusion has a similar defining equation to a stochastic Loewner-Kufarev equation. More precisely, let $(g_t)_{0\le t\le T}$ be univalent functions on $\mathbb{D}_0$ satisfying our stochastic L-K equation. Then the inverse process $g_t^{-1} : g_t(\mathbb{D}_0) \to \mathbb{D}_0$ obeys

$$\mathrm{d}g_t^{-1}(z) = -g_t^{-1}(z)\Big\{\frac{\mathrm{d}t}{\alpha_0} + \sum_{k=1}^{\infty} g_t^{-1}(z)^k \frac{\mathrm{d}Z_t^k}{\alpha_k}\Big\}.$$

Let σ_t be Malliavin's canonic diffusion. Then the stochastic process $\sigma_t(1)$ on S^1 verifies

$$\mathrm{d}\sigma_t(1) = -\sigma_t(1)\Big\{\frac{\gamma}{2}\mathrm{d}t + \sum_{k=1}^{\infty} \frac{-i\mathbf{Re}\big(\sigma_t(1)^k \mathrm{d}\widetilde{Z}_t^k\big)}{\sqrt{hk + \frac{c}{12}(k^3-k)}}\Big\},$$

where $\gamma := \sum_{k=1}^{\infty}\{hk + \frac{c}{12}(k^3-k)\}^{-1}$ and $\widetilde{Z}_t^k := x_t^{(k,1)} - ix_t^{(k,2)}$.

Comparison with Malliavin's canonic diffusion. II

Proof.
Firstly, we have

$$0 = \mathrm{d}\big(g_t(g_t^{-1}(z))\big) = \big(\mathrm{d}g_t\big)(g_t^{-1}(z)) + \underbrace{g_t'(g_t^{-1}(z)) \circ \mathrm{d}g_t^{-1}(z)}_{=g_t'(g_t^{-1}(z))\mathrm{d}g_t^{-1}(z)},$$

so that (with putting $w := g_t^{-1}(z)$),

$$(\mathrm{d}g_t)(z) = -\frac{\big(\mathrm{d}g_t\big)(w)}{g_t'(w)} = -\frac{wg_t'(w)\left\{\mathrm{d}X_t^0 + \sum_{k=1}^{\infty} w^k \mathrm{d}X_t^k\right\}}{g_t'(w)}$$

$$= -g_t^{-1}(z)\left\{\mathrm{d}X_t^0 + \sum_{k=1}^{\infty} g_t^{-1}(z)^k \mathrm{d}X_t^k\right\}.$$

Comparison with Malliavin's canonic diffusion. III

On the other hand, with defining $\theta_0 : \mathrm{Diff}_+(S^1) \to \mathbb{R}$ by

$$\theta_0(\sigma) := \theta(\sigma(1)), \quad \sigma \in \mathrm{Diff}_+(S^1),$$

we see easily that

$$\big(c_k\theta_0\big)(\sigma) = \cos(k\theta_0(\sigma)), \quad \big(s_k\theta_0\big)(\sigma) = \sin(k\theta_0(\sigma)).$$

Then we have

$$\begin{aligned}
\mathrm{d}\theta_0(\sigma_t) &= \sum_{k=1}^{\infty} \frac{\big(c_k\theta_0\big)(\sigma_t) \circ \mathrm{d}x_t^{(k,1)} + \big(s_k\theta_0\big)(\sigma_t) \circ \mathrm{d}x_t^{(k,2)}}{\sqrt{hk + \frac{c}{12}(k^3 - k)}} \\
&= \sum_{k=1}^{\infty} \frac{\cos(k\theta_0(\sigma_t)) \circ \mathrm{d}x_t^{(k,1)} + \sin(k\theta_0(\sigma_t)) \circ \mathrm{d}x_t^{(k,2)}}{\sqrt{hk + \frac{c}{12}(k^3 - k)}} \\
&= \sum_{k=1}^{\infty} \frac{\cos(k\theta_0(\sigma_t))\mathrm{d}x_t^{(k,1)} + \sin(k\theta_0(\sigma_t))\mathrm{d}x_t^{(k,2)}}{\sqrt{hk + \frac{c}{12}(k^3 - k)}},
\end{aligned}$$

Comparison with Malliavin's canonic diffusion. IV

where in the last equality, the stochastic contraction appeared. Writing $\widetilde{Z}_t^k := x_t^{(k,1)} - i x_t^{(k,2)}$, the above equation can be written as

$$\mathrm{d}\theta_0(\sigma_t) = \sum_{k=1}^{\infty} \frac{\mathrm{Re}\big(\mathrm{e}^{ik\theta_0(\sigma_t)}\mathrm{d}\widetilde{Z}_t^k\big)}{\sqrt{hk + \frac{c}{12}(k^3 - k)}},$$

so that

$$\begin{aligned} \mathrm{d}\,\overbrace{\mathrm{e}^{i\theta_0(\sigma_t)}}^{=\sigma_t(1)} &= i\mathrm{e}^{i\theta_0(\sigma_t)} \circ \mathrm{d}\theta_0(\sigma_t) \\ &= -\frac{1}{2}\mathrm{e}^{i\theta_0(\sigma_t)}\gamma \mathrm{d}t + i\mathrm{e}^{i\theta_0(\sigma_t)} \sum_{k=1}^{\infty} \frac{\mathrm{Re}\big(\mathrm{e}^{ik\theta_0(\sigma_t)}\mathrm{d}\widetilde{Z}_t^k\big)}{\sqrt{hk + \frac{c}{12}(k^3 - k)}}, \end{aligned}$$

where $\gamma := \sum_{k=1}^{\infty}\{hk + \frac{c}{12}(k^3 - k)\}^{-1}$. $\square$

Comparison with Malliavin's canonic diffusion. V

Remark.

- ▷ Although it is known that the complexification of $\mathrm{Diff}_+(S^1)$ does not exist, one can write down heuristically the corresponding SDE on the complexification, and a heuristic calculation shows that the canonic diffusion "on" the non-existing complexification of $\mathrm{Diff}_+(S^1)$ is a solution to our stochastic Loewner-Kufarev equation.

- ▷ Existence- and uniqueness- properties of solutions to our stochastic Loewner-Kufarev equation have not been established yet. Instead of that, we shall focus on a hierarchy of the equation.

A Hierarchical Solution to Stochastic Loewner-Kufarev Equation.

A Hierarchy of Stochastic L-K Equation. I

Proposition

Let a family of holomorphic $g_t : \mathbb{D}_0 \to \mathbb{C}$ *satisfy the SDE*

$$dg_t(z) = zg_t'(z)\Big\{dX_t^0 + \sum_{k=1}^{\infty} z^k dX_t^k\Big\}, \quad g_0(z) \equiv z \in \mathbb{D}_0$$

where $X_t^0 = \alpha_0^{-1} t$ *and* $X_t^k = \alpha_k^{-1} Z_t^k$ *for* $k \geq 1$. *We parametrize* g_t *as*

$$g_t(z) = C(t)\big(z + c_1(t)z^2 + c_2(t)z^3 + c_3(t)z^4 + \cdots\big).$$

Then we have

$$\begin{cases} dC(t) = C(t)dX_t^0, \\ dc_1(t) = dX_t^1 + c_1(t)dX_t^0, \\ dc_n(t) = dX_t^n + \displaystyle\sum_{k=1}^{n-1}(k+1)c_k(t)dX_t^{n-k} + nc_n(t)dX_t^0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 2. \end{cases}$$

A Hierarchy of Stochastic L-K Equation. II

Although it is not known whether or not the equation for g_t has a solution, we notice that the previous system consists of linear stochastic differential equations with constant coefficients and hence this system can be integrated and admits a unique strong solution $(C(t), c_1(t), c_2(t), \cdots)$.

Definition

We call a sequence $(C(t), c_1(t), c_2(t), \cdots)$ of $\mathbb{C}$-valued stochastic processes satisfying the previous system of SDEs a **hierarchical solution** to the stochastic L-K equation

$$\mathrm{d}g_t(z) = z g_t'(z)\Big\{\mathrm{d}X_t^0 + \sum_{k=1}^{\infty} z^k \mathrm{d}X_t^k\Big\}, \quad g_0(z) \equiv z \in \mathbb{D}_0.$$

In the deterministic case, the dynamics of the corresponding $(C(t), c_1(t), c_2(t), \cdots)$ is studied by Vasiliev and his coauthors.

Integration by Parts Formula.

Kirillov-Neretin Polynomials.

Let

$$\mathrm{Aut}(\mathcal{O}) := \Big\{ c_0(z + \sum_{k=1}^{\infty} c_k z^{k+1}) \in \mathbb{C}[\![z]\!] : c_0 \neq 0 \Big\}.$$

We regard $(c_1, c_2, \cdots)$ as a coordinate on $\mathrm{Aut}(\mathcal{O})/\mathbb{C}c_0$.
For each holomorphic function
$f(z) = f'(0)(z + \sum_{k=1}^{\infty} c_k z^{k+1}) \in \mathrm{Aut}(\mathcal{O})$ and constant c, the **Kirillov-Neretin polynomials** (of conformal weight $h = 0$) $P_n(c_1, \cdots, c_n)$ $n \geq 0$ are defined by

$$\sum_{n=0}^{\infty} P_n(c_1, \cdots, c_n) z^n = \frac{cz^2}{12} \mathcal{S}_f(z),$$

where $\mathcal{S}_f(z) = \frac{f'''(z)}{f''(z)} - \frac{3f''(z)^2}{2f'(z)^2}$ is the Schwarzian derivative of f.

Kirillov-Yuriev's representation of Virasoro.

The positive part of an well-known representation $\{-z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}\}_{n\in\mathbb{Z}}$ of Witt algebra lifts on $\mathrm{Aut}(\mathcal{O})/\mathbb{C}c_0$ by

$$z^{n+1}f'(z) = L_n f, \quad n \geq 1$$

where with setting $\partial_n = \dfrac{\partial}{\partial c_n}$,

- $L_n = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}$, $n \geq 1$.

Kirillov-Yuriev (1988) showed $\{L_n\}_{n\geq 1}$ can be extended to a collection $\{L_n\}_{n\in\mathbb{Z}}$ of operators on $\mathrm{Aut}(\mathcal{O})$ satisfying the commutation relation of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}.$$

Integration by Parts I

Theorem

Let $(C(t), c_1(t), c_2(t), \cdots)$ be a hierarchical solution to the stochastic L-K equation and set $P_n(t) := P_n(c_1(t), \cdots, c_n(t))$ for each $n \geq 0$. Then for each polynomial $F(c_1, c_2, \cdots)$, we have

$$
\begin{aligned}
&\mathbf{E}\big[(L_n F)(c_1(t), c_2(t), \cdots)\big] \\
&= \mathbf{E}\big[F(c_1(t), c_2(t), \cdots) \times \underbrace{\overline{\begin{pmatrix} \textit{combination of } c_1(t) \\ \textit{and Neretin polynomials} \\ \textit{up to n-th order} \end{pmatrix}}}_{=:\mathrm{div}_{\mathbf{P}} L_n}\big]
\end{aligned}
$$

for a.a. t and $n \geq 1$. Where the divergence terms may include the stochastic integrals of $(P_k(s))_{0 \leq s \leq t}$, $k = 1, 2, \cdots, n$.

Integration by Parts II

For example, with setting $\alpha_0^{-1} = 0$ and $\gamma_k := \frac{c}{12}(k^3 - k)$ for simplicity, it holds that $\dfrac{\gamma_2}{\alpha_2}\mathrm{div}_{\mathbf{P}} L_2 = P_2(t)$ which is equal to $\mathcal{S}_{g_t}(0)$ if $g_t(z) = C(t)(z + \sum_{k=1}^{\infty} c_k(t) z^{k+1})$ converges and then, with setting $\alpha_2 := \frac{c}{12}\gamma_2$ (> 0 if $c \neq 0$), the last theorem is stating roughly that

$$\text{“}\int_{\text{Loops}} (L_2 F)(\mathcal{L})\, \mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big) = \frac{c}{12} \int_{\text{Loops}} F(\mathcal{L}) \overline{\mathcal{S}_{g_t}(0)}\, \mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big)\text{”},$$

for any “polynomial function” F. Finally, we remark that this is close to one of the properties required to a Kontsevich-Suhov's conjectural measure on loop space.

Thank you for your attention.