An Integration by Parts on "Space of Loops"

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We study a probability measure on "space of loops" induced by a (alternate) Loewner-Kufarev equation

$$(1) \qquad \frac{\partial g_t}{\partial t}(z) = z g_t'(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{i\theta} + z}{\mathrm{e}^{i\theta} - z} \nu_t(\theta) \mathrm{d}\theta, \quad g_0(z) \equiv z \in \mathbb{D}_0,$$

where $\mathbb{D}_0$ stands for the unit disk in the complex plane and the input ν_t is set to be

$$\text{“}\nu_t(\theta) = \alpha_0^{-1} t + \sum_{k=1}^{\infty} \alpha_k^{-1} \big\{ \dot{B}_t^{(k,1)} \cos(k\theta) + \dot{B}_t^{(k,2)} \sin(k\theta) \big\}\text{”}.$$

Here, $(B_t^{(k,1)}, B_t^{(k,2)})$, $k \geq 1$ are infinitely many independent two-dimensional Brownian motions and $(\alpha_k)_{k \geq 0}$ are positive real numbers. With calculating the right-hand-side in (1), we are motivated to consider the following SDE, which we call a stochastic Loewner-Kufarev equation:

$$(2) \qquad \mathrm{d}g_t(z) = z g_t'(z) \big\{ \mathrm{d}X_t^0 + \sum_{k=1}^{\infty} z^k \mathrm{d}X_t^k \big\}, \quad g_0(z) \equiv z \in \mathbb{D}_0,$$

where $X_t^0 = \alpha_0^{-1} t$, $X_t^k = \alpha_k^{-1} Z_t^k$ and Z_t^k's are infinitely many independent complex Brownian motions. We need to take α_k's so that the right-hand-side in (2) converges.

The main result is the following.

Theorem 1. *Malliavin's canonic diffusion "on"* $\mathrm{Diff}_+(S^1)$ *(see* [2]*) has a similar defining equation to a stochastic Loewner-Kufarev equation. More precisely, let* $(g_t)_{0 \leq t \leq T}$ *be univalent function on* $\mathbb{D}_0$ *satisfying the equation (2). Then the inverse process* $g_t^{-1} : g_t(\mathbb{D}_0) \to \mathbb{D}_0$ *obeys*

$$\mathrm{d}g_t^{-1}(z) = -g_t^{-1}(z) \Big\{ \frac{\mathrm{d}t}{\alpha_0} + \sum_{k=1}^{\infty} g_t^{-1}(z)^k \frac{\mathrm{d}Z_t^k}{\alpha_k} \Big\}.$$

Let σ_t *be Malliavin's canonic diffusion. Then the stochastic process* $\sigma_t(1)$ *on* S^1 *verifies*

$$\mathrm{d}\sigma_t(1) = -\sigma_t(1) \Big\{ \frac{\gamma}{2} \mathrm{d}t + \sum_{k=1}^{\infty} \frac{-i \mathrm{Re}\big(\sigma_t(1)^k \mathrm{d}\widetilde{Z}_t^k\big)}{\sqrt{hk + \frac{c}{12}(k^3 - k)}} \Big\},$$

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where $\gamma := \sum_{k=1}^{\infty}\{hk+\frac{c}{12}(k^3-k)\}^{-1}$, $\widetilde{Z}_t^k := x_t^{(k,1)} - i x_t^{(k,2)}$ *and* $(x_t^{(k,1)}, x_t^{(k,2)})$ *are infinitely many independent two-dimensional Brownian motions.*

Existence- and uniqueness- properties of solutions to (2) have not been established yet. Instead of that, we shall focus on a hierarchy of (2).

Proposition 2. *Let a family of holomorphic* $g_t : \mathbb{D}_0 \to \mathbb{C}$ *satisfy the equation (2). We parametrize* g_t *as*

$$g_t(z) = C(t)\big(z + c_1(t)z^2 + c_2(t)z^3 + c_3(t)z^4 + \cdots\big).$$

Then we have

$$\tag{3} \begin{cases} \mathrm{d}C(t) = C(t)\mathrm{d}X_t^0, \\ \mathrm{d}c_1(t) = \mathrm{d}X_t^1 + c_1(t)\mathrm{d}X_t^0, \\ \mathrm{d}c_n(t) = \mathrm{d}X_t^n + \displaystyle\sum_{k=1}^{n-1}(k+1)c_k(t)\mathrm{d}X_t^{n-k} + nc_n(t)\mathrm{d}X_t^0, \\ \hfill n \geq 2. \end{cases}$$

We notice that the system (3) consists of linear stochastic differential equations with constant coefficients and hence this system can be integrated and admits a unique strong solution $(C(t), c_1(t), c_2(t), \cdots)$. We call such the sequence a *hierarchical solution* to (2).

For each $c \in \mathbb{R}$, the Kirillov-Neretin polynomials $P_n(c_1, \cdots, c_n)$, $n \geq 0$ are defined by $\sum_{n=0}^{\infty} P_n(c_1, \cdots, c_n)z^n = \frac{cz^2}{12}\mathcal{S}_f(z)$, where $\mathcal{S}_f(z)$ is the Schwarzian derivative of $f(z) = f'(0)(z + \sum_{k=1}^{\infty} c_k z^{k+1})$. Define $L_n := \frac{\partial}{\partial c_n} + \sum_{k=1}^{\infty}(k+1)c_k\frac{\partial}{\partial c_{n+k}}$ for $n \geq 1$.

Theorem 3. *Let* $(C(t), c_1(t), c_2(t), \cdots)$ *be a hierarchical solution to (2) and set* $P_n(t) := P_n(c_1(t), \cdots, c_n(t))$ *for each* $n \geq 0$*. Then for each polynomial* $F(c_1, c_2, \cdots)$*, we have*

$$\begin{aligned} &\mathbf{E}\big[(L_nF)(c_1(t), c_2(t), \cdots)\big] \\ &= \mathbf{E}\Big[F(c_1(t), c_2(t), \cdots) \times \underbrace{\overline{\begin{pmatrix} \textit{combination of } c_1(t) \\ \textit{and Neretin polynomials} \\ \textit{up to } n\textit{-th order} \end{pmatrix}}}_{=:\mathrm{div}_{\mathbf{P}}L_n}\Big] \end{aligned}$$

for a.a. t *and* $n \geq 1$*. Where the divergence terms may include the stochastic integrals of* $(P_k(s))_{0 \leq s \leq t}$, $k = 1, 2, \cdots, n$.

For example, with setting $\alpha_0^{-1} = 0$ and $\gamma_k := \frac{c}{12}(k^3 - k)$ for simplicity, it holds that $\dfrac{\gamma_2}{\alpha_2}\mathrm{div}_{\mathbf{P}}L_2 = P_2(t)$ which is equal to $\mathcal{S}_{g_t}(0)$ if $g_t(z) = C(t)(z + \sum_{k=1}^{\infty} c_k(t)z^{k+1})$ converges and then, with setting $\alpha_2 := \frac{c}{12}\gamma_2$ (> 0 if $c \neq 0$), the last theorem is stating roughly that

$$\text{“}\int_{\text{Loops}} (L_2F)(\mathcal{L})\,\mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big) = \frac{c}{12}\int_{\text{Loops}} F(\mathcal{L})\overline{\mathcal{S}_\phi(0)}\,\mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big)\text{”},$$

for any "polynomial function" F (cf. the equations (2.31) and (2.35) in [1]).

References

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[2] Malliavin, P. (1999) *The canonic diffusion above the diffeomorphism group of the circle.* C. R. Acad. Sci. Paris Sér. I Math. 329, no. 4, 325-329. MR 1713340