研究集会「確率解析とその周辺」

平成 26 年度科学研究費補助金基盤研究 (B) 課題番号 24340023「無限次元空間上の確率解析」(研究代表者：会田 茂樹), 平成 26 年度科学研究費補助金基盤研究 (S) 課題番号 23224003「非線形発展方程式の凝縮現象と解の構造」(研究代表者：堤 誉志雄) の援助を受けて, 表記の研究集会を以下の要領で開催致しますのでご案内申し上げます。

日時： 2014 年 10 月 14 日（火）10:20 ～ 10 月 16 日（木）16:00

場所： 東北大学理学部数理科学記念館 (川井ホール)
〒980-8578 宮城県仙台市青葉区荒巻字青葉 6 番 3 号

ホームページ： `https://www.math.kyoto-u.ac.jp/probability/sympo/sa14/`

—プログラム—

10 月 14 日 (火)

10:20〜11:00　道工 勇 (Isamu Dôku)　(埼玉大学)　[40 分]
Unbiased estimator and some integral equations

11:10〜11:50　中島 誠 (Makoto Nakashima)　(筑波大学)　[40 分]
Stochastic heat equation arising from a certain branching systems in random environment

12:00〜12:40　泉 優行 (Yuki Izumi)　(九州大学)　[40 分]
L^p-solutions of backward stochastic differential equations and their Malliavin derivatives

12:40〜14:00　昼休み

14:00〜14:50　コハツ-ヒガ アルトゥーロ (Arturo Kohatsu-Higa)　(立命館大学)　[50 分]
The parametrix as a stochastic method

15:00〜15:50　土屋 貴裕 (Takahiro Tsuchiya)　(会津大学)　[50 分]
A note on convergence rates for stability problems of SDEs

16:10〜16:50　石渡 聡 (Satoshi Ishiwata)　(山形大学)　[40 分]
A central limit theorem for non-symmetric random walks on crystal lattices

17:00〜17:40　正宗 淳 (Jun Masamune)　(東北大学)　[40 分]

Selfadjoint extensions of a Schrödinger-type operator

10 月 15 日 (水)

10:20〜11:00　和田 正樹 (Masaki Wada)　(東北大学)　[40 分]

Large time asymptotics for Feynman-Kac functionals of symmetric stable processes

11:10〜11:50　天羽 隆史 (Takafumi Amaba)　(立命館大学)　[40 分]

An integration by parts on space of loops

12:00〜12:40　藤田 安啓 (Yasuhiro Fujita)　(富山大学)　[40 分]

A proof of L^p-logarithmic Sobolev inequality via several approximations

12:40〜14:00　昼休み

14:00〜15:00　Massimiliano Gubinelli　(Universite Paris-Dauphine)　[60 分]

Singular stochastic PDEs and paracontrolled distributions

15:10〜16:00　廣島 文生 (Fumio Hiroshima)　(九州大学)　[50 分]

Stochastic renormalization in QFT

16:20〜17:00　永沼 伸顕 (Nobuaki Naganuma)　(東北大学)　[40 分]

Exact convergence rate of the Wong-Zakai approximation to RDEs driven by Gaussian rough paths

17:10〜17:50　稲浜 譲 (Yuzuru Inahama)　(名古屋大学)　[40 分]

Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion

10 月 16 日 (木)

10:20〜11:00　中津 智則 (Tomonori Nakatsu)　(立命館大学)　[40 分]

Integration by parts formulas concerning maxima of some SDEs with applications

11:10〜11:50　植村 英明 (Hideaki Uemura)　(愛知教育大学)　[40 分]

Identification of noncausal functions from the stochastic Fourier coefficients without the aid of a Brownian motion

12:00〜12:40　伊藤 悠 (Yu Ito)　(京都大学)　[40 分]

Differential equations driven by rough paths: An approach via fractional calculus

12:40〜14:00　昼休み

14:00〜15:00　Massimiliano Gubinelli　(Universite Paris-Dauphine)　[60 分]

Regularisation by noise in PDEs

15:10〜16:00　福泉 麗佳 (Reika Fukuizumi)　(東北大学)　[50 分]

Vortex solutions in Bose-Einstein condensation

世話人　会田 茂樹（東北大学大学院理学研究科）
重川 一郎（京都大学大学院理学研究科）
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楠岡 誠一郎（東北大学大学院理学研究科）

Unbiased Estimator and Some Integral Equations

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Let $D_0 := \mathbb{R}^3 \setminus \{0\}$, and we put $\mathbb{R}_+ := [0, \infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, we use the symbol $\alpha \cdot \beta$ for the inner product, and we define $e_x := x/|x|$ for every $x \in D_0$. We consider the following deterministic nonlinear integral equation:

$$e^{\lambda t|x|^2} u(t,x) = u_0(x) + \frac{\lambda}{2}\int_0^t \mathrm{d}s\ e^{\lambda s|x|^2} \int p(s,x,y;u)n(x,y)\mathrm{d}y + \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2} f(s,x)\mathrm{d}s, \qquad \text{for} \quad \forall (t,x) \in \mathbb{R}_+ \times D_0. \tag{1}$$

Here $u \equiv u(t,x)$ is an unknown function : $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \to \mathbb{C}^3$ is the initial data such that $u(t,x)|_{t=0} = u_0(x)$. Moreover, $f(t,x) : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$ for each $x \in D_0$. The integrand p in (1) is given by

$$p(t,x,y;u) = u(t,y) \cdot e_x\{u(t,x-y) - e_x(u(t,x-y)\cdot e_x)\}. \tag{2}$$

Suppose that the integral kernel $n(x,y)$ is bounded and measurable with respect to $\mathrm{d}x \times \mathrm{d}y$. On the other hand, we consider a Markov kernel K described below. For every $z \in D_0$, $K_z(\mathrm{d}x, \mathrm{d}y)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel k is given by $k(x,y) = i|x|^{-2}n(x,y)$, then we define K_z as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$\iint h(x,y)K_z(\mathrm{d}x, \mathrm{d}y) = \int h(x, z-x)k(x,z)\mathrm{d}x. \tag{3}$$

We shall start with defining of a branching Markov chain $\{\xi_n\}_n$. First of all, we put

$$\mathbb{D} := \bigoplus_{n=0}^{\infty} D_0^n, \qquad D_0^0 = \varDelta \notin D_0^n \quad (n = 1, 2, \dots),$$

and let D_0 be equipped with the σ-field $\mathcal{G}$ generated by a natural σ-algebra $\mathcal{B}(D_0)$ and a single-point set $\{\Delta\}$, where Δ is called an absorbing state in the theory of Markov processes. For each $x \in D_0$ we define a probability measure $p(x, \mathrm{d}y) \in \mathcal{P}(\mathbb{D}, \mathcal{G})$ such that $p_n(x, A) := p(x, A)$ for $A \subset D_0^n$ and $g(x) = p(x, \{\Delta\})$ as a measurable function. Here $p(x, \mathrm{d}y)$ is called a transition function of Markov chain $\{\xi_n\}$. We assume that for every $A \in \mathcal{G}$, $p(x, A)$ is a $\mathcal{B}(D_0)$-measurable function.

THEOREM. *Let* $\{\xi_k\}_{k=0}^{\infty}$ *be a Markov chain with a phase space* $(\mathbb{D}, \mathcal{G})$ *and with a transition function* $p(x, \mathrm{d}y)$. *Then there exist a suitable sequence* $\{\tau_m\}$ *of random variables and a proper functional* $M^*(\xi) = M^*(\tau_m, \{\xi_k\})$ *of* $\{\tau_m\}$ and $\{\xi_k\}$ *such that a random quantity* $\zeta = M^*(\xi)$ *is a realizable unbiased estimator of the solutions to the integral equation* (1), i.e., *in other words, the function*

$$u(t, x) := \hat{E}_{t,x}[M^*(\xi)] = \hat{E}_{t,x}[M^*(\tau, \{\xi_k\})] \tag{4}$$

satisfies the equation (1), *where* $\hat{E}_{t,x}$ *is the expectation with respect to a probability measure* $Q_{t,x}$ *on* $(\mathbb{D}, \mathcal{G})$.

For brevity's sake we illustrate the typical case by a simple example, to see what on earth the above-mentioned unbiased estimator $\zeta = M^*(\xi)$ is really like. Let us now consider a simple case of $\Xi(\varphi_{p(m)}(\xi))$ involved with $\varphi_{p(m)}(\xi)$ and $\varphi_{p(m')}(\xi)$ with pivoting $\xi_{p(m'')}$. Actually, $\varphi_{p(m)}(\xi)$ and $\varphi_{p(m')}(\xi)$ are functional (having an explicit form) of Markov chain $\{\xi_n\}$. Then we have immediately

$$\begin{aligned} & M^*(\xi) \equiv M^*(\tau_m, \{\xi_n\}) = \Xi(\varphi_{p(m)}(\xi)) \\ & = i\alpha(\xi_{p(m'')}) \times \beta(\varphi_{p(m)}(\xi), \xi_{p(m'')}) \times \gamma(\xi_{p(m'')}, \varphi_{p(m')}(\xi)) \\ & = \frac{i \sum_{j=1}^{3} \varphi_k^j(\xi) \xi_{k''}^j}{\sqrt{(\xi_{k''}^1)^2 + (\xi_{k''}^2)^2 + (\xi_{k''}^3)^2}} \times \left\{ \frac{\sum_{j=1}^{3} \xi_{k''}^j \varphi_{k'}^j(\xi)}{(\xi_{k''}^1)^2 + (\xi_{k''}^2)^2 + (\xi_{k''}^3)^2} \cdot \xi_{k''} - \varphi_{k'}(\xi) \right\} \end{aligned}$$

where we put $p(m) = k$, $p(m') = k'$ and $p(m'') = k''$, and both particles with labels m and m' belong to the same ℓ-th generation of descendants since we have $|m| = |m'| = \ell$ when $|m''| = \ell - 1$.

Stochastic heat equation arising from a certain branching systems in random environment

Makoto Nakashima
University of Tsukuba, Graduate School of Pure and Applied Sciences*

In this talk, we will consider the stochastic heat equations on the line which have been studied for four decades. Especially, we will construct a non-negative solution to a certain stochastic heat equation by using a branching systems in random environment.

1 Stochastic heat equation

In this talk, we consider the stochastic heat equations as follows:

$$\frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta X_t(x) + a(X_t(x))\dot{W}(t,x), \tag{1.1}$$

where W is a time-space white noise and a is a continuous function with $a(0) = 0$.

The study of stochastic heat equation was started around 1970's. In particular, the existence and the uniqueness of the strong solution to (1.1) are known if a is Lipschitz continuous [8] et.al.

Also, the existence of the solution to (1.1) are verified for more general a under some initial conditions [7]. On the other hand, the uniqueness of solutions to (1.1) are very difficult problem attacked by many mathematicians [4, 3] et. al.

The stochastic heat equations (1.1) appear as some limit process. One of the most famous examples is a one-dimensional super-Brownian motion which is a measure-valued process arising as a scaling limit of some critical branching Brownian motion or branching random walks.

2 Super-Brownian motion

Before giving a definition of super-Brownian motion, we recall the branching random walks.*

Definition 1. *Branching random walks are defined as follows:*

(1) There are particles at $x_1, \cdots, x_{M_N} \in \mathbb{Z}^d$ at time 0.

(2) The particles at time n choose a nearest neighbor site independently and uniformly, and move there.

(3) Then, each of them independently splits into two particles with probability $\frac{1}{2}$ or vanishes with probability $\frac{1}{2}$.

Remark: The total number at time n, B_n, is a critical Galton-Watson process.

We set a measure-valued process $\{X_t^{(N)}\}$ as follows: For every Borel set A

$$X_0^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{M_N} \delta_{x_i/N^{1/2}}(dx),$$

$$X_t^{(N)}(A) = \frac{1}{N}\sharp\{\text{particles locates in } N^{1/2}A \text{ at time } \lfloor Nt \rfloor\}.$$

Then, we have the following theorem:

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*In this talk, we consider the most simple case.

Theorem 2. *([9, 1]) If $X_0^{(N)} \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$, then $\{X_\cdot^{(N)}\}$ weakly converges to a measure valued process X_t as $N \to \infty$.*

Moreover,[2, 6] if $d = 1$, then X_t is absolutely continuous with respect to the Lebesgue measure for any $t >$ a.s. and its density $X_t(x)$ is the unique non-negative weak solution to the stochastic heat equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}(t,x), \quad \lim_{t\to 0} X_t(x)dx = X_0(dx).$$

3 Main result

We construct a solution to (1.1) with $a(u) = \sqrt{u}$ from a certain branching system in random environment.

Theorem 3. *([5]) For any $X_0 \in \mathcal{M}_F(\mathbb{R})$, there exists the unique, weak, and non-negative solution to the stochastic heat equation*

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x) + X_t(x)^2}\dot{W}(t,x), \lim_{t\to\infty} X_t(x)dx = X_0(dx).$$

Remark: Mytnik gave a remark on the above construction in his paper.

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L^p Solutions of Backward Stochastic Differential Equations and their Malliavin Derivatives

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Backward Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $W = (W_t)_{0\le t\le T}$ be an n-dimensional standard Brownian motion defined on the space, and $(\mathcal{F}_t)_{0\le t\le T}$ be the Browinian filtration augmented by all P-null sets. $T > 0$ represents a terminal time.

We consider the following d dimensional stochastic differential equation:

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad Y_T = \xi,$$

which is often rewritten in the form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s^* dW_s, \quad 0 \le t \le T,$$

where ξ is an $\mathcal{F}_T$-measurable random variable which represents a terminal condition, $f\colon [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \to \mathbb{R}^d$ with progressive measurablity for each element in $\mathbb{R}^d \times \mathbb{R}^{n\times d}$. These type of SDEs are called backward SDEs (BSDEs for short).

For $p > 1$, an L^p solution of a BSDE is a pair (Y, Z), composed of $(\mathcal{F}_t)$-adapted continuous process and $(\mathcal{F}_t)$-progressively measurable process respectively, which satisfies the BSDE and

$$E\left[\sup_{0\le t\le T} |Y_t|^p\right] + E\left[\left(\int_0^T |Z_t|^2 dt\right)^{\frac{p}{2}}\right] < \infty.$$

L^p solutions and their Malliavin derivatives

Under appropriate assumptions on ξ and f, it is known that BSDE with respect to ξ and f has unique L^p solution.

Then El Karoui, Peng and Quenez [1] showed the L^p ($p \ge 2$) solution of BSDE is differentiable in Malliavin's sense. In addition, an important property between Y and Z is given; $Z_t = D_t Y_t$, where $D_u = \frac{d}{dt}\big|_{t=u} \nabla$ and ∇ represents the Malliavin derivative operator.

Malliavin derivatives of Wiener functionals take values on Hilbert spaces. They are specifically Hilbert-Schmidt operators. Therefore it is useful to consider BSDEs on Hilbert spaces when trying to differentiate many times. In this talk, we deal with BSDEs on Hilbert spaces and introduce some results on higher order Malliavin differentiability of solutions.

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The parametrix as a stochastic method

Arturo Kohatsu-Higa (Ritsumeikan University)

We will introduce the parametrix as a stochastic method and we will describe possible applications of these representations in simulation problems.

A NOTE ON CONVERGENCE RATES FOR STABILITY PROBLEMS OF SDES

TAKAHIRO TSUCHIYA

Consider the following sequence of one-dimensional stochastic differential equations

$$X_n(t) = X_n(0) + \int_0^t b_n(X_n(s))ds + \int_0^t \sigma_n(X_n(s))dW_s, \tag{1}$$

and consider the solution X given by

$$X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW_s, \tag{2}$$

where $\{W_s\}_{s\geq 0}$ is a Wiener process and $b_n : \mathbb{R} \to \mathbb{R}$ and $\sigma_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$ are coefficients which tend to b and σ respectively in some sense, as $n \to \infty$. The convergence of the sequence $\{X_n\}_{n\in\mathbb{N}}$ to X named as stability problems was introduced by Stroock and Varadhan in [11] to solve the martingale problems for unbounded coefficients b and σ. On the other hand, the stability approach in the sense of *strong* to the continuous diffusion coefficients, typically Hölder continuous of exponent $(\frac{1}{2}+\gamma)$ for $0 \leq \gamma \leq 1/2$, has been further developed by Kawabata-Yamada [6]. The study of strong solution for irregular coefficients was treated by Nakao in [9] and also Zvonkin and Krylov in [7] where σ is bounded below by a positive constant and is of bounded variation on any compact interval. Then the result of [9] were extended to by Le Gall in [8], who proved the stability problems on the diffusion coefficients are positive and squared finite quadratic variation.

In another important development, the "rate" of convergence of the *Euler-Maruyama schemes* to the solution of (2) was drawn primarily from the paper Deelstra and Delbaen in [1] and their results has been considerably generalized by Gyöngy and Rásonyi in [3]: The convergence rate of *Euler-Maruyama schemes* with the diffusion coefficients satisfying $(\frac{1}{2}+\gamma)$-Hölder continuous and the suitable drift coefficients in L^1 is bounded by $n^{-\gamma}$ where $0 < \gamma \leq 1/2$, however, that is bounded by $(\log n)^{-1}$ in the case where the diffusion coefficients $(1/2)$-Hölder continuous, $\gamma = 0$.

The result suggests that the rate of convergence may depend also on the modulus continuity of diffusion coefficients. Therefore the goal of this research is to estimate the strong convergence rate of *stability problems*. To be more precious, let us consider the following drift-less stability problems,

$$X_n(t) - X_n(0) = \int_0^t \sigma_n(X_n(s))dW_s, \quad X(t) - X(0) = \int_0^t \sigma(X_n(s))dW_s,$$

for $t \geq 0$ and $X_n(0) \equiv X(0)$.

As a first result, the strong convergence rate is given of the stability problem when the coefficients satisfy the modulus continuity; σ and σ_n are $\left(\frac{1}{2}+\gamma\right)$-Hölder continuous and σ_n converges to σ uniformly. Then, there exists a positive constant C_1 such that

$$\mathbb{E}(|X(t) - X_n(t)|) \leq \begin{cases} C_1 n^{-\gamma} & (0 < \gamma \leq 1/2) \\ C_1 (\log n)^{-1} & (\gamma = 0). \end{cases}$$

Since the coefficient may be discontinuous under the following Nakao-Le Gall condition, a typically example is Skew Brownian motions [4], it seems to be very interesting to investigate the rate of convergence of the stability problems under the condition:

Definition 1 (Nakao-Le Gall condition)**.** *We say that a real valued function σ satisfies the Nakao-Le Gall condition and write $\sigma \in \mathcal{C}_{NL}(\epsilon, f)$ if σ satisfies the following statements:*

(*i*) *There exists a positive real number ϵ such that*

$$\epsilon \leq \sigma(x)$$

holds for any x in $\mathbb{R}$.

(*ii*) *There exists a monotone increasing function f such that*

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)|$$

holds for every x and y in $\mathbb{R}$.

(*iii*) *In addition, the function f is bounded on $\mathbb{R}$,*

$$\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)| < \infty.$$

In this presentation, the strong convergence rate of the Nakao-Le Gall condition will be given as the local time argument plays an important role to estimate. Thanks to the extended local time expression for rotation invariant and α-stable processes Z given by Fitzsimmons and Getoor [2], K. Yamada [12], and see also [10]. Then the result will be extended to the rotation invariant and index α process Z driven stochastic differential equations;

$$X_n(t) - X_n(0) = \int_0^t \sigma_n(X_n(s)) dZ_s, \quad X(t) - X(0) = \int_0^t \sigma(X_n(s)) dZ_s,$$

for $t \geq 0$ and $X_n(0) \equiv X(0)$.

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A central limit theorem for non-symmetric random walks on crystal lattices

Satoshi Ishiwata (Yamagata University) *

This talk is based on a joint work with Hiroshi Kawabi(Okayama Univ.) and Motoko Kotani (Tohoku Univ.).

A locally finite, connected oriented graph $X = (V, E)$ is called *crystal lattice* if X is an abelian covering graph of a finite graph $X_0 = (V_0, E_0)$. We denote by $\Gamma \simeq \mathbb{Z}^d$ the covering transformation group. Our interest is the long time behavior of the transition probability

$$p(n, x, y) = \sum_{\substack{(e_1, e_2, \ldots, e_n) \in C_{x,n} \\ t(e_n) = y}} p(e_1)p(e_2) \cdots p(e_n)$$

given by a 1-step transition probability $p : E \to [0, 1]$ satisfying

$$\sum_{e \in E_x} p(e) = 1, \quad p(e) + p(\overline{e}) > 0, \quad \forall \sigma \in \Gamma,\, p(\sigma e) = p(e).$$

There are many results of this problem under some various settings. See Spitzer [10], Lawler [9] and references therein. Our study is motivated by the following local central limit theorem (LCLT) presented by Sunada [11]:

Theorem 0.1 *Suppose that the random walk is irreducible with period K. Then*

$$p(n, x, y) \sim \frac{K \mathrm{vol}(\mathrm{Alb}^{\Gamma})m(y)}{(2\pi n)^{d/2}} \exp\left(-\frac{\|\Phi(y) - \Phi(x) - n\rho_{\mathbb{R}}(\gamma_p)\|^2}{2n}\right),$$

where m is the (lift of) normalized invariant measure on X_0, γ_p is the homological direction, $\rho_{\mathbb{R}}$ is the canonical surjective homomorphism from $\mathrm{H}_1(X_0, \mathbb{R})$ to $\Gamma \otimes \mathbb{R}$, $\Phi : X \to \Gamma \otimes \mathbb{R}$ is the modified harmonic realization, defined by

$$\forall x \in V, \quad \Delta\Phi(x) := \sum_{e \in E_x} p(e) \left(\Phi(o(x)) - \Phi(t(e))\right) = \rho_{\mathbb{R}}(\gamma_p),$$

and $\| \cdot \|$ is the Albanese metric on $\Gamma \otimes \mathbb{R}$, induced by

$$\begin{array}{ccc}
\Gamma \otimes \mathbb{R} & \longleftarrow & \mathrm{H}_1(X_0, \mathbb{R}) \\
\updownarrow & & \updownarrow \\
\mathrm{Hom}(\Gamma, \mathbb{R}) & \hookrightarrow & \mathrm{H}^1(X_0, \mathrm{R}) \simeq \mathcal{H}^1(X_0).
\end{array}$$

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Here $\mathcal{H}^1(X_0)$ is the space of modified harmonic 1-forms difined by

$$\forall x \in V_0, \quad \delta\omega(x) + \langle \gamma_p, \omega \rangle = 0$$

equipped with a canonical inner product defined by

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle = \sum_{e \in E_0} p(e)\omega_1(e)\omega_2(e)m(o(e)) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle.$$

See also .[2], [3], [4], [5], [6], [7], [8], [12].

It is natural to ask the weak convergence of the sequence of law of the probability measure of the random walk on X. In this talk we give two canonical weak convergences.

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Selfadjoint extensions of a Schrödinger-type operator

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Let $M = (M, g)$ be a smooth Riemannian manifold without boundary, not necessarily geodesically complete. We consider a Schrödinger-type operator

$$L = \Delta + V$$

where $\Delta = \operatorname{div} \circ \nabla$ is the Laplace-Beltrami operator and V is a real-valued continuous function on M. By Green's formula, L is a symmetric operator on the space of smooth functions with compact support $C_0^\infty(M)$ in $L^2 = L^2(M; dv_g)$, that is,

$$(Lu, v) = (u, Lv), \qquad \forall u, v \in C_0^\infty(M)$$

where (u, v) stands for the L^2-inner product of u and v. In this talk we will discuss several problems regarding with selfadjoint extensions of L; in particular, the essential selfadjointness, i.e., L has unique selfadjoint extension, as well as the Markov uniqueness by which we mean that there exists unique selfadjoint extension $\hat{L}$ satisfying the Markov property:

$$0 \le u \le 1 \quad \Rightarrow \quad 0 \le T_t u \le 1, \quad \forall t > 0 \tag{1}$$

where $\{T_t\}_{t\ge 0}$ is the L^2-semigroup generated by $\hat{L}$. Roughly speaking, (1) means that the system should not increase the energy with time $t > 0$, and together with an additional assumption[1], $\{T_t\}_{t\ge0}$ is a transition probability of a Markov process (actually, a Hunt process) on M due to Fukushima's theorem.

Therefore, these two problems are to determine the possible physical models in quantum mechanics and Markov processes respectively, or more precisely, to study the negligibility of the singular sets of the space by these systems.

Our starting point is M.P. Gaffney [3] proving that Δ has unique Markov extension provided M is geodesically complete. This result has been extended to various directions: P.R. Chernoff [4] and R.S. Strichartz [6] showed the essential self-adjointness of Δ for geodesically complete manifolds. See [2, 5, 8, 7, 9] for results on geodesically incomplete manifolds. Another direction is to study the behavior of V and the geometry of M so that L is essentially selfadjoint (see, e.g., [1] and the reference within).

[1]The domain $\mathcal{F} = \mathrm{Dom}(\sqrt{-L})$ of the energy $\mathcal{E}$ satisfies that $\mathcal{F} \cap C_\infty(M)$ is dense in $\mathcal{F}$ and $C_\infty(M)$ with respect to $\|u\| = \sqrt{\mathcal{E}(u,u) + (u,u)}$ and sup-norm, respectively.

The main discussion of the talk will be devoted to some new results along those lines for geodesically incomplete manifolds when the singular points enjoy a certain symmetry. Time permitting, we will also address a recent result for

$$L = \Delta + X + V$$

where $X \in \Gamma(TM)$ answering the same question but it's non-symmetric counter part as a generalization of [10].

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Large time asymptotics for Feynman-Kac functionals of symmetric stable processes

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Let $\{X_t\}$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and denote by $(\mathscr{E}, \mathscr{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d)$. We assume $\alpha < d$, transience of $\{X_t\}$ and denote the Green kernel by $G(x,y)$. Let μ be a positive Radon smooth measure satisfying Green-tightness and define the Schrödinger form $\mathscr{E}^\mu$ by $\mathscr{E}^\mu(u,v) = \mathscr{E}(u,v) - \langle u,v\rangle_\mu \equiv \langle -\mathscr{L}^\mu u, v\rangle$. Denoting by A_t^μ the positive continuous additive functional in the Revuz correspondence with μ, we have

$$\int_{\mathbb{R}^d} p^\mu(t,x,y)dy = \mathbb{E}_x[\exp(A_t^\mu)]. \tag{1}$$

Here $p^\mu(t,x,y)$ is the fundamental solution of the equation $\partial u/\partial t = \mathscr{L}^\mu u$. We call the right hand side of (1) *Feynman-Kac functional*. In this talk, we consider the large time asymptotics for $\mathbb{E}_x[\exp(A_t^\mu)]$. This is a jointly work with Professor Masayoshi Takeda.

We define the spectral bottom of the time changed process for $\{X_t\}$ by μ as follows:

$$\lambda(\mu) = \inf\{\mathscr{E}(u,u) \mid u \in \mathscr{F}_e, \quad \langle u,u\rangle_\mu = 1\},$$

where $\mathscr{F}_e$ is the extended Dirichlet space. Note that $\lambda(\mu)$ represents the smallness of μ. If $\lambda(\mu) > 1$, μ is said to be *subcritical*. Takeda [3] showed that μ is subcritical if and only if $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[\exp(A_\infty^\mu)] < \infty$. Moreover, if μ is of 0-order finite energy integral, this condition is also equivalent to the stability of fundamental solution, i.e. $p^\mu(t,x,y)$ admits the same two-sided estimates as the transition density function of $\{X_t\}$ up to positive multiple constants ([5]).

If $\lambda(\mu) < 1$, μ is said to be *supercritical*. The supercriticality of μ is equivalent to

$$C(\mu) := -\inf\{\mathscr{E}^\mu(u,u) \mid u \in \mathscr{F}, \quad \langle u,u\rangle = 1\} > 0$$

and this is the principal eigenvalue of $\mathscr{L}^\mu$. Via Fukushima's ergodic theorem, Takeda [4] showed $\mathbb{E}_x[\exp(A_t^\mu)] \sim c_1 h(x)\exp(C(\mu)t)$ where $h(x)$ is the eigenfunction corresponding to the principal eigenvalue.

If $\lambda(\mu) = 1$, μ is said to be *critical*. In this case $C(\mu) = 0$ and the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$ is not exponential. Simon [2] and Cranston, Koralov et al. [1] treated the same problem for Brownian motion. They gave a concrete growth order of $\mathbb{E}_x[\exp(A_t^\mu)]$ depending on d for absolutely continuous μ with some additional conditions. For the proof, they first gave the asymptotic expansion of the β-order resolvent $G_\beta(x,y)$ as $\beta \to 0$ using the Hankel functions. The Schrödinger resolvent $\{G^\mu_\beta\}$ is expressed through G_β and the resolvent equation. Since it follows that $\mathbb{E}_x[\exp(A^\mu_t)] = 1 + \int_0^t P^\mu_s \mu ds$ for the Schrödinger semigroup $\{P^\mu_s\}$, their results follow via Tauberian theorem and the behavior of $G^\mu_\beta \mu$ as $\beta \to 0$.

In our framework, we impose only compactness on μ and thus need some improvements of their methods. First, we cannot express the resolvent kernel of the α-stable processes through special functions. The expression of the transition density function and some calculations enable us to obtain

$$G_\beta(x,y) = G_0(x,y) - c_1 k(\beta)|x-y|^{(2\alpha-d)\wedge 0} + E_\beta(x,y).$$

Here $k(\beta)$ is a function depending on d/α and $E_\beta(x,y)$ has smaller order than $k(\beta)$. Secondary, we consider the time changed process by μ for β-killed process of $\{X_t\}$ to obtain the representation of $G^\mu_\beta \mu$, since μ is not necessarily absolutely continuous. The Green operator of this process is given by $f \to \int_{\mathbb{R}^d} G_\beta(\cdot,y) f(y)\mu(dy)$, and a compact operator on $L^2(\mu)$. Thus, we can apply the perturbation theory and conclude that $k(\beta)G^\mu_\beta \mu$ converges $\mathscr{E}$-weakly as $\beta \to 0$. Since P^μ_ε admits Green-tight integral kernel, we can strengthen this convergence to pointwise one and obtain the following result:

Theorem 1. *(Takeda-W. 2014)*
Suppose $\{X_t\}$ is the transient, rotationally invariant α-stable process and μ is a critical measure with compact support. As $t \to \infty$, Feynman-Kac functional satisfies

$$\mathbb{E}_x[\exp(A^\mu_t)] \sim \begin{cases} c_1 h_0(x) t^{d/\alpha - 1} & (1 < d/\alpha < 2), \\ c_2 h_0(x) t/\log t & (d/\alpha = 2), \\ c_3 h_0(x) t & (d/\alpha > 2), \end{cases}$$

where $h_0(x)$ is the ground state of $\mathscr{E}^\mu$.

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An Integration by Parts on "Space of Loops"

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We study a probability measure on "space of loops" induced by a (alternate) Loewner-Kufarev equation

$$\frac{\partial g_t}{\partial t}(z) = z g_t'(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(\theta) d\theta, \quad g_0(z) \equiv z \in \mathbb{D}_0, \tag{1}$$

where $\mathbb{D}_0$ stands for the unit disk in the complex plane and the input ν_t is set to be

$$\text{“}\nu_t(\theta) = \alpha_0^{-1} t + \sum_{k=1}^{\infty} \alpha_k^{-1} \big\{ \dot{B}_t^{(k,1)} \cos(k\theta) + \dot{B}_t^{(k,2)} \sin(k\theta) \big\}\text{”}.$$

Here, $(B_t^{(k,1)}, B_t^{(k,2)})$, $k \geq 1$ are infinitely many independent two-dimensional Brownian motions and $(\alpha_k)_{k \geq 0}$ are positive real numbers. With calculating the right-hand-side in (1), we are motivated to consider the following SDE, which we call a stochastic Loewner-Kufarev equation:

$$d g_t(z) = z g_t'(z) \big\{ dX_t^0 + \sum_{k=1}^{\infty} z^k dX_t^k \big\}, \quad g_0(z) \equiv z \in \mathbb{D}_0, \tag{2}$$

where $X_t^0 = \alpha_0^{-1} t$, $X_t^k = \alpha_k^{-1} Z_t^k$ and Z_t^k's are infinitely many independent complex Brownian motions. We need to take α_k's so that the right-hand-side in (2) converges.

The main result is the following.

Theorem 1. *Malliavin's canonic diffusion "on"* $\mathrm{Diff}_+(S^1)$ *(see* [2]*) has a similar defining equation to a stochastic Loewner-Kufarev equation. More precisely, let* $(g_t)_{0 \leq t \leq T}$ *be univalent function on* $\mathbb{D}_0$ *satisfying the equation (2). Then the inverse process* $g_t^{-1} : g_t(\mathbb{D}_0) \to \mathbb{D}_0$ *obeys*

$$d g_t^{-1}(z) = -g_t^{-1}(z) \Big\{ \frac{dt}{\alpha_0} + \sum_{k=1}^{\infty} g_t^{-1}(z)^k \frac{dZ_t^k}{\alpha_k} \Big\}.$$

Let σ_t *be Malliavin's canonic diffusion. Then the stochastic process* $\sigma_t(1)$ *on* S^1 *verifies*

$$d\sigma_t(1) = -\sigma_t(1) \Big\{ \frac{\gamma}{2} dt + \sum_{k=1}^{\infty} \frac{-i \mathrm{Re}\big(\sigma_t(1)^k d\widetilde{Z}_t^k\big)}{\sqrt{hk + \frac{c}{12}(k^3 - k)}} \Big\},$$

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where $\gamma := \sum_{k=1}^{\infty}\{hk+\frac{c}{12}(k^3-k)\}^{-1}$, $\widetilde{Z}_t^k := x_t^{(k,1)} - ix_t^{(k,2)}$ *and* $(x_t^{(k,1)}, x_t^{(k,2)})$ *are infinitely many independent two-dimensional Brownian motions.*

Existence- and uniqueness- properties of solutions to (2) have not been established yet. Instead of that, we shall focus on a hierarchy of (2).

Proposition 2. *Let a family of holomorphic* $g_t : \mathbb{D}_0 \to \mathbb{C}$ *satisfy the equation (2). We parametrize* g_t *as*

$$g_t(z) = C(t)\big(z + c_1(t)z^2 + c_2(t)z^3 + c_3(t)z^4 + \cdots\big).$$

Then we have

$$(3)\qquad \begin{cases} \mathrm{d}C(t) = C(t)\mathrm{d}X_t^0, \\ \mathrm{d}c_1(t) = \mathrm{d}X_t^1 + c_1(t)\mathrm{d}X_t^0, \\ \mathrm{d}c_n(t) = \mathrm{d}X_t^n + \displaystyle\sum_{k=1}^{n-1}(k+1)c_k(t)\mathrm{d}X_t^{n-k} + nc_n(t)\mathrm{d}X_t^0, \\ \hfill n \geq 2. \end{cases}$$

We notice that the system (3) consists of linear stochastic differential equations with constant coefficients and hence this system can be integrated and admits a unique strong solution $(C(t), c_1(t), c_2(t), \cdots)$. We call such the sequence a *hierarchical solution* to (2).

For each $c \in \mathbb{R}$, the Kirillov-Neretin polynomials $P_n(c_1, \cdots, c_n)$, $n \geq 0$ are defined by $\sum_{n=0}^{\infty} P_n(c_1, \cdots, c_n)z^n = \frac{cz^2}{12}\mathcal{S}_f(z)$, where $\mathcal{S}_f(z)$ is the Schwarzian derivative of $f(z) = f'(0)(z + \sum_{k=1}^{\infty} c_k z^{k+1})$. Define $L_n := \frac{\partial}{\partial c_n} + \sum_{k=1}^{\infty}(k+1)c_k\frac{\partial}{\partial c_{n+k}}$ for $n \geq 1$.

Theorem 3. *Let* $(C(t), c_1(t), c_2(t), \cdots)$ *be a hierarchical solution to (2) and set* $P_n(t) := P_n(c_1(t), \cdots, c_n(t))$ *for each* $n \geq 0$*. Then for each polynomial* $F(c_1, c_2, \cdots)$*, we have*

$$\begin{aligned} &\mathbf{E}\big[(L_nF)(c_1(t), c_2(t), \cdots)\big] \\ &= \mathbf{E}\big[F(c_1(t), c_2(t), \cdots) \times \underbrace{\overline{\begin{pmatrix} \textit{combination of } c_1(t) \\ \textit{and Neretin polynomials} \\ \textit{up to } n\textit{-th order} \end{pmatrix}}}_{=:\mathrm{div}_{\mathbf{P}}L_n}\big] \end{aligned}$$

for a.a. t *and* $n \geq 1$*. Where the divergence terms may include the stochastic integrals of* $(P_k(s))_{0\leq s\leq t}$*,* $k = 1, 2, \cdots, n$*.*

For example, with setting $\alpha_0^{-1} = 0$ and $\gamma_k := \frac{c}{12}(k^3-k)$ for simplicity, it holds that $\dfrac{\gamma_2}{\alpha_2}\mathrm{div}_{\mathbf{P}}L_2 = P_2(t)$ which is equal to $\mathcal{S}_{g_t}(0)$ if $g_t(z) = C(t)(z + \sum_{k=1}^{\infty} c_k(t)z^{k+1})$ converges and then, with setting $\alpha_2 := \frac{c}{12}\gamma_2$ (> 0 if $c \neq 0$), the last theorem is stating roughly that

$$\text{“}\int_{\text{Loops}}(L_2F)(\mathcal{L})\,\mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big) = \frac{c}{12}\int_{\text{Loops}} F(\mathcal{L})\overline{\mathcal{S}_\phi(0)}\,\mathbf{P}\big(g_t(S^1) \in \mathrm{d}\mathcal{L}\big)\text{”},$$

for any “polynomial function” F (cf. the equations (2.31) and (2.35) in [1]).

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A PROOF OF L^p–LOGARITHMIC SOBOLEV INEQUALITY VIA SEVERAL APPROXIMATIONS

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This talk is based on [7].
For a smooth enough function $f \geq 0$ on $\mathbb{R}^n$, we define the entropy of f with respect to the Lebesgue measure by

$$\mathrm{Ent}(f) = \int f(x) \log f(x) dx - \int f(x) dx \ \log \int f(x) dx.$$

In this talk, the integral without its domain is always understood as the one over $\mathbb{R}^n$, and we interpret that $0 \log 0 = 0$.

Let $p \geq 1$. We denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions f on $\mathbb{R}^n$ such that f and $|Df|$ (the Euclidean length of the gradient Df of f) are in $L^p(\mathbb{R}^n)$. For $f \in W^{1,p}(\mathbb{R}^n)$, the following L^p–logarithmic Sobolev inequality was shown for $p = 2$ by [11], $p = 1$ by [10], and $1 < p < n$ by [6]:

$$\mathrm{Ent}(|f|^p) \leq \frac{n}{p} \int |f(x)|^p dx \ \log \left(L_p \frac{\displaystyle\int |Df(x)|^p \, dx}{\displaystyle\int |f(x)|^p \, dx} \right). \tag{1}$$

Here,

$$L_p = \begin{cases} \dfrac{p}{n} \left(\dfrac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left(\dfrac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n \frac{p-1}{p} + 1\right)} \right)^{p/n}, & p > 1, \\ \dfrac{1}{n} \pi^{-1/2} \left[\Gamma\left(\dfrac{n}{2} + 1 \right) \right]^{1/n}, & p = 1. \end{cases} \tag{2}$$

This is the best possible constant satisfying (1) for $1 \leq p < n$ (cf. [1, 6]).

For a general $p > 1$, with a deep insight, Gentil [9, Theorem 1.1] gave inequality (1) by using a hypercontractivity inequality for the unique viscosity solution to the Cauchy problem of a Hamilton-Jacobi equation. However, his proof for inequality (1) is valid for a special class of functions f in $W^{1,p}(\mathbb{R}^n)$.

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Our aim of this talk is to bridge the gap in the proof of [9, Theorem 1.1] and provide a supplementary proof of inequality (1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. The strategy of our proof is the following:

First, we show (1) for $f \in W^{1,p}(\mathbb{R}^n)$ such that

$$f \in C^1(\mathbb{R}^n),\ 0 < f \leq 1 \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n. \tag{3}$$

Second, we approximate $f \in W^{1,p}(\mathbb{R}^n)$ by a sequence of functions satisfying (3) by several steps. This is the key point to derive (1). An important tool is the following Fatou–type inequality: if a family $\{f_\epsilon\}_{0<\epsilon<1}$ of nonnegative and measurable functions on $\mathbb{R}^n$ approximates a function f in some sense, then

$$\liminf_{\epsilon \to 0+} \int f_\epsilon(x)^p \log f_\epsilon(x) dx \geq \int f(x)^p \log f(x) dx. \tag{4}$$

Finally, by using these approximations, we show that L^p–logarithmic Sobolev inequality (1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$.

I express my hearty appreciation to Ivan Gentil for his encouragement.

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Massimiliano Gubinelli (Universite Paris-Dauphine)

Lecture 1: Singular stochastic PDEs and paracontrolled distributions

Non-linear evolution problems perturbed by singular noise sources arise naturally as scaling limits of certain microscopic evolutions or homogenisation problems. The parabolic anderson model, the Kardar-Parisi-Zhang equation and the stochastic quantization equation are examples of such systems. Solving (or even giving a meaning to) these equations require a detailed understanding of the propagation of the stochastic perturbations via the non-linear evolution. I will explain how ideas and tools from harmonic analysis can be useful in this analysis and in the related problem of studying the convergence of the microscopic models to their scaling limits.

Lecture 2: Regularisation by noise in PDEs

It can happen that randomly perturbing a PDE can lead to better properties for the solutions. Random perturbation have usually sample paths which are very irregular and it seems that is this irregularity to play a key role in the regularisation effect. I will introduce a deterministic and quantitative notion of irregularity for functions of one variable and show how it can be used to analyse the behaviour of linear and non-linear PDEs modulated by such irregular perturbations. The following situations will be considered: linear transport equations, non-linear Schrödinger equations and the KdV equation.

Stochastic renormalization in QFT

Fumio Hiroshima (Kyushu University)

This is a joint work with M. Gubinelli and J. Lorinczi [JFA 2014]. By using a stochastic method we renormalize UV cutoff imposed on a scalar quantum field model. This is an alternative of the method by Edward Nelson who renormalized UV cutoff by a functional analysis method. Physical and mathematical interpretations of the renormalized term is given and some application are discussed.

Exact convergence rate of the Wong-Zakai approximation to RDEs driven by Gaussian rough paths

Nobuaki Naganuma *

Abstract

We consider a solution to a stochastic differential equation (SDE) driven by a Gaussian process in the sense of rough differential equation (RDE) and the Wong-Zakai approximation to the solution. We give an upper bound of the error of the Wong-Zakai approximation. We also show that the upper bound is optimal in a particular case.

1 Introduction

The rough path theory originated from Lyons gives a framework that allows us to deal with differential equations driven by rough signals rigorously. After Lyons' work, many researchers apply it to SDEs. In context of SDEs, the rough path theory plays a crucial role in order to study differential equations with rougher driving signals than Brownian motion; for example, fractional Brownian motions and more general Gaussian processes.

A key step to consider SDEs in the rough path theory is to construct rough paths associated to the drivers. Coutin-Qian showed an existence of a rough path associated to a Gaussian process under the condition so-called the Coutin-Qian condition (Defintion 1) and Friz-Victoir proved an existence under more mild conditions on its covariance function. Once we construct the rough paths associated to the drivers of SDEs, we obtain solutions to SDEs automatically with help of the rough path theory. Moreover, we can obtain a pathwise estimate for the difference of two solutions to SDEs which have different drivers; by using the local Lipschitz continuity of the Itô-Lyons map, we see that the difference of the solutions inherit from the difference of the drivers. However, we need to make an effort to obtain a probabilistic error bounds. Since the Lipschitz constant appeared in the Itô-Lyons map is a random variable, we need to consider integrability of it. The integrability is proved by [CLL13, FR13, BFRS13].

In this talk, combing the integrability of the Lipschitz constant stated above and the estimates of two rough paths (Theorem 2), we obtain the exact convergence rates of the approximations to SDE (Theorem 3).

2 Main results

Let $X = (X^1, \dots, X^d)$ be a continuous, centered d-dimensional Gaussian process with independent and identically distributed components. We assume that X satisfies the following the Coutin-Qian condition:

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Definition 1. We say that X satisfies the Coutin-Qian conditions for $0 < \lambda < 1$ if there exists a positive constant $\mathcal{C}_\lambda$ such that

$$\boldsymbol{E}\left[(X_t^\alpha - X_s^\alpha)^2\right] \le \mathcal{C}_\lambda |t-s|^{2\lambda} \text{ for any } 0 < s,t < 1,$$
$$\left|\boldsymbol{E}\left[(X_{s+\epsilon}^\alpha - X_s^\alpha)(X_{t+\epsilon}^\alpha - X_t^\alpha)\right]\right| \le \mathcal{C}_\lambda |t-s|^{2\lambda-2}\epsilon^2 \text{ for any } 0 < \epsilon < |t-s|.$$

We define the m-th dyadic polygonal approximation $X(m)$ to X by

$$X(m)_t = (X_{\tau_k^m} - X_{\tau_{k-1}^m})2^m(t - \tau_{k-1}^m) + X_{\tau_{k-1}^m}$$

for $\tau_{k-1}^m \le t \le \tau_k^m$, where $\tau_k^m = k2^{-m}$. Denote by $\boldsymbol{X}(m)$ the natural rough path associated to $X(m)$. It is known that there exists a limit rough path $\boldsymbol{X}$ in $(G\Omega_p(\mathbf{R}^d), \rho_{p\text{-var}})$ under the Coutin-Qian condition for $1/4 < \lambda \le 1/2$ and $\lambda p > 1$. Here $G\Omega_p(\mathbf{R}^d)$ is the space of geometric rough paths and $\rho_{p\text{-var}}$ is the (inhomogeneous) metric which is defined by

$$\rho_{p\text{-var}}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \max_{1 \le \ell \le \lfloor p \rfloor} \rho_{p\text{-var}}^{(\ell)}(\boldsymbol{x}, \tilde{\boldsymbol{x}}),$$
$$\rho_{p\text{-var}}^{(\ell)}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_k=1} \left(\sum_{l=1}^k |\boldsymbol{x}_{\tau_{l-1}\tau_l}^\ell - \tilde{\boldsymbol{x}}_{\tau_{l-1}\tau_l}^\ell|_{(\mathbf{R}^d)^{\otimes \ell}}^{p/\ell} \right)^{\ell/p}.$$

Under this setting, we obtain the following:

Theorem 2. *Assume that X satisfies the Coutin-Qian condition for $1/3 < \lambda < 1/2$. Then*

$$\boldsymbol{E}[|\rho_{p\text{-}var}(\boldsymbol{X}, \boldsymbol{X}(m))|^r]^{1/r} \le C 2^{-m(2\lambda - 1/2)}$$

for any $r \ge 1$ and $p > 1/(1/2 - \lambda)$.

Theorem 3. *Let $\sigma \in C_{bdd}^\infty(\mathbf{R}^e; \mathbf{R}^d \otimes \mathbf{R}^e)$. Assume that X satisfies the Coutin-Qian condition for $1/3 < \lambda < 1/2$. Consider the solutions to SDEs*

$$\begin{cases} dY_t = \sigma(Y_t)\, dX_t, \\ Y_0 = y_0 \in \mathbf{R}^e, \end{cases} \qquad \begin{cases} dY(m)_t = \sigma(Y(m)_t)\, dX(m)_t, \\ Y(m)_0 = Y_0 \in \mathbf{R}^e. \end{cases}$$

Then, for any $r \ge 1$, there exists a positive constant C independent of m such that satisfy

$$\boldsymbol{E}\left[\left(\sup_{0 \le t \le 1} |Y_t - Y(m)_t|\right)^r\right]^{1/r} \le C 2^{-m(2\lambda - 1/2)}.$$

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Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion

Yuzuru Inahama (Nagoya University)

Abstract: We study a stochastic differential equation in the sense of rough path theory driven by fractional Brownian rough path with Hurst parameter H $(1/3 < H \le 1/2)$ under the ellipticity assumption at the starting point. In such a case, the law of the solution at a fixed time has a kernel, i.e., a density function with respect to Lebesgue measure. (See [1]). In this paper we prove a short time *off-diagonal* asymptotic expansion of the kernel under mild additional assumptions. Our main tool is Watanabe's distributional Malliavin calculus developped in [2]. Unlike some other works on asymptotics for SDEs driven by fBm, our RDE (1) has a drift term. This makes the asymptotic expansion quite comlicated. Note also that when $H = 1/2$, SDE (1) is just a Stratonovich SDE driven by the usual Brownian motion. Therefore, our result can be regards as a generaliztion of Watanabe [2].

Let $(w_t)_{t\ge0} = (w_t^1, \ldots, w_t^d)_{t\ge0}$ be the d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (1/3, 1/2]$. Let $V_i : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞, that is, V_i is a bounded smooth function with bounded derivatives of all order $(0 \le i \le d)$. We consider the following (random) rough differential equation (RDE) driven by fractional Brownian rough path, i.e., the natural lift of fBm (w_t);

$$dy_t = \sum_{i=1}^d V_i(y_t)dw_t^i + V_0(y_t)dt \qquad \text{with} \qquad y_0 = a \in \mathbf{R}^n. \tag{1}$$

We will sometimes write $y_t = y_t(a)$ etc. to make explicit the dependence on a.

First, we assume the ellipticity of the coefficient of (1) at the starting point $a \in \mathbf{R}^n$.

(A1): The set of vectors $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbf{R}^n$.

Under Assumption **(A1)**, the law of the solution y_t has a density $p_t(a, a')$ with respect to the Lebesgue measure da' on $\mathbf{R}^n$ for any $t > 0$. Let $\mathcal{H} = \mathcal{H}^H$ be the Cameron-Martin space of fBm (w_t). For $\gamma \in \mathcal{H}$, we denote by $\phi_t^0 = \phi_t^0(\gamma)$ be the solution of the following Young ODE;

$$d\phi_t^0 = \sum_{i=1}^d V_i(\phi_t^0)d\gamma_t^i \qquad \text{with} \qquad \phi_0^0 = a \in \mathbf{R}^n.$$

Set, for $a \neq a'$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

If we assume **(A1)** for all a, this set $K_a^{a'}$ is not empty. If $K_a^{a'}$ is not empty, it is a Hilbert submanifold of $\mathcal{H}$. It is known that $\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$. Now we introduce the following assumption;

(A2): $\bar\gamma \in K_a^{a'}$ which minimizes $\mathcal{H}$-norm exists uniquely.

In the sequel, $\bar\gamma$ denotes the minimizer in Assumption **(A2)**. We also assume that the Hessian of $\|\cdot\|_{\mathcal{H}}^2/2$ is not so degenerate at $\bar\gamma$ in the following sense.

(A3): At $\bar\gamma$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathcal{H}}^2/2$ is strictly larger than $\mathrm{Id}_{\mathcal{H}^H}/2$ in the form sense. More precisely, If $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $f(0) = \bar\gamma$ and $f'(0) \neq 0$, then $(d/du)^2|_{u=0}\|f(u)\|_{\mathcal{H}}^2/2 > 0$.

Now,we introduce several index sets for the exponent of the small parameter $\varepsilon := t^H > 0$, which will be used in the asymptotic expansion. Unlike in many preceding papers, index sets in this paper are not (a constant multiple of) $\mathbf{N} = \{0, 1, 2, \dots\}$ and are quite complicated.

Set $\Lambda_1 = \{n_1 + \frac{n_2}{H} \mid n_1, n_2 \in \mathbf{N}\}$. We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ all the elements of Λ_1 in increasing order. Several smallest elements are explicitly given as follows; $\kappa_1 = 1, \quad \kappa_2 = 2, \quad \kappa_3 = \frac{1}{H}, \quad \kappa_4 = 3, \quad \kappa_5 = 1 + \frac{1}{H}, \dots$ As usual, using the scale invariance (i.e., self-similarity) of fBm, we will study the scaled version of (1). From its explicit form, one can easily see why Λ_1 appears.

We also set $\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\} = \{0, 1, \frac{1}{H} - 1, 2\,\frac{1}{H}, 3 \dots\}$ and $\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1\}\} = \{0, \frac{1}{H} - 2, 1, \frac{1}{H} - 1, 2, \dots\}$. Next we set

$$\Lambda_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda_2\}.$$

In the sequel, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ stands for all the elements of Λ_3 in increasing order. Similarly,

$$\Lambda_3' = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda_2'\}.$$

In the sequel, $\{0 = \rho_0 < \rho_1 < \rho_2 < \cdots\}$ stands for all the elements of Λ_3' in increasing order. Finally, $\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}$. We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ all the elements of Λ_4 in increasing order.

Below we state two main results of ours, which are basically analogous to the corresponding ones in Watanabe [2]. However, there are some differences. First, the exponents on the shoulder of t are not (a constant multiple of) natural numbers. Second, cancellation of "odd terms" as in p. 20 and p. 34, [2] does not happen in general in our case. (If the drift term in RDE (1) is zero or if $H = 1/2$, then this kind of cancellation takes place).

The following is a short time asymptotic expansion of the diagonal of the kernel function. This is much easier than the off-diagonal case.

Theorem 1 *Assume* **(A1)**. *Then, the diagonal of the kernel $p(t, a, a)$ admits the following asymptotics as $t \searrow 0$;*

$$p(t, a, a) \sim \frac{1}{t^{nH}} \big(c_0 + c_{\nu_1} t^{\nu_1 H} + c_{\nu_2} t^{\nu_2 H} + \cdots\big)$$

for certain real constants c_{ν_j} $(j = 0, 1, 2, \dots)$. Here, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ are all the elements of Λ_3 in increasing order.

We also have off-diagonal short time asymptotics of the kernel function. This is our main result.

Theorem 2 *Assume* **(A1)**–**(A3)**. *Then, we have the following asymptotic expansion as $t \searrow 0$;*

$$p(t, a, a') \sim \exp\Big(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2t^{2H}}\Big) \frac{1}{t^{nH}} \big\{c_{\lambda_0} + c_{\lambda_1} t^{\lambda_1 H} + c_{\lambda_2} t^{\lambda_2 H} + \cdots\big\}$$

for certain real constants c_{λ_j} $(j = 0, 1, 2, \dots)$. Here, $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ are all the elements of Λ_4 in increasing order.

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Integration by parts formulas concerning maxima of some SDEs with applications

Tomonori Nakatsu (Ritsumeikan University)

1 Introduction

In this talk, firstly, we shall deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \tag{1}$$

where $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0, \infty)\}$ denotes a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We will consider discrete time maximum and continuous time maximum which are defined by $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$ and $M_T := \max_{0 \le t \le T} X_t$, respectively, where the time interval $[0, T]$ and the time partition $0 \le t_1 < \cdots < t_n = T$, $n \ge 2$ are fixed.

Secondly, we will deal with the following d-dimensional SDE,

$$Z_t^i = z_0^i + \int_0^t V_0^i(Z_s) ds + \sum_{j=1}^d \int_0^t V_j^i(Z_s) \circ dW_s^j, \ 1 \le i \le d,$$

where $V_j^i : \mathbb{R}^d \to \mathbb{R}$, $0 \le j \le d, 1 \le i \le d$ are measurable functions and $\circ dW^j$ denotes the Stratonovich integral with respect to a d-dimensional standard Brownian motion $\{W_t = (W_t^1, \cdots, W_t^d), t \in [0, \infty)\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$. For this d-dimensional SDE, we shall consider the random variable defined by $\hat{M}_T := \max\{Z_T^1, \cdots, Z_T^d\}$, where $T > 0$ is fixed.

In this talk, we say that an integration by parts (IBP) formula for random variables F and G holds if there exists a random variable $H(F; G)$ such that $E^P[\varphi'(F)G] = E^P[\varphi(F)H(F; G)]$ holds for any φ in a class of C^1 functions, where $E^P[\cdot]$ denotes the expectation with respect to a probability measure P. The IBP formula is usually used to obtain expressions and upper bounds of the probability density function of F by taking $G = 1$. Meanwhile, in finance, IBP formulas play an important role in order to compute the risks of financial products, called greeks (see [1], for example).

Our goal is to prove IBP formulas for M_T^n, M_T and $\hat{M}_T$, in addition, to obtain the expressions and upper bounds of their probability density functions by means of the IBP formulas.

2 Main results

Assumption (A)

(A1) For $t \in [0, \infty)$, $b(t, \cdot), \sigma(t, \cdot) \in C_b^2(\mathbb{R}; \mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t, \cdot)$ and $\sigma(t, \cdot)$ do not depend on t.

(A2) There exists $c > 0$ such that

$$|\sigma(t, x)| \ge c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0, \infty)$.

Theorem 1. *Assume (A). Let $G \in \mathbb{D}^{1,\infty}$ and assume $t_1 > 0$. Then there exists a random variable $H_T^n(G)$ such that $H_T^n(G)$ belongs to $L^p(\Omega, \mathcal{F}, P)$ for any $p \ge 1$, and*

$$E^P \left[\varphi'(M_T^n)G\right] = E^P \left[\varphi(M_T^n) H_T^n(G)\right] \tag{2}$$

holds for any $\varphi \in C_b^1(\mathbb{R}; \mathbb{R})$.

Remark 1. *In the case that* $t_1 = 0$*, (2) in Theorem 1 is valid for any* $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ *whose support is included in* (x_0, ∞)*.*

Assumption (A)'

We assume that the diffusion coefficient of (1) is of the form $\sigma(t,x) = \sigma_1(t)\sigma_2(x)$ and the following assumption.

(A1)' For $t \in [0,\infty)$, $b(t,\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ do not depend on t.

(A2)' $\sigma_1(\cdot) \in C_b^0([0,\infty);\mathbb{R})$ and there exists $c_1 > 0$ such that $|\sigma_1(t)| \geq c_1$ for any $t \in [0,\infty)$.

(A3)' $\sigma_2(\cdot) \in C_b^3(\mathbb{R};\mathbb{R}_+)$(respectively, $C_b^3(\mathbb{R};\mathbb{R}_-)$), $x \mapsto \sigma_2(x)$ is increasing (respectively, decreasing) and there exists $c_2 > 0$ such that $|\sigma_2(x)| \geq c_2$ for any $x \in \mathbb{R}$.

Theorem 2. *Assume (A)'. Let* $G \in \mathbb{D}^{1,\infty}$ *and* $a_0 > x_0$ *be fixed arbitrarily. Then there exists a random variable* $H_T(G,a_0)$ *such that* $H_T(G,a_0)$ *belongs to* $L^p(\Omega,\mathcal{F},P)$ *for any* $p \geq 1$*, and*

$$E^P\left[\varphi'(M_T)G\right] = E^P[\varphi(M_T)H_T(G,a_0)]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ *whose support is included in* (a_0,∞)*.*

Define

$$a(x) := VV^T(x),$$

for $x \in \mathbb{R}^d$, where V^T is the transpose matrix for V.

Assumption (B)

(B1) For each $1 \leq i,j \leq d$, $V_j^i(\cdot) \in C_b^2(\mathbb{R}^d;\mathbb{R})$.

(B2) There exists $c > 0$ such that

$$\langle \xi, a(x)\xi \rangle \geq c|\xi|^2,$$

holds for any $x, \xi \in \mathbb{R}^d$.

(B3) Vector fields $V_1, \cdots, V_d$ are commutative, that is

$$[V_i,V_j](x) = [V_j,V_i](x), 1 \leq i,j \leq d$$

hold for any $x \in \mathbb{R}^d$, where we have defined the Lie bracket by $[V_i,V_j](x) := \nabla V_j V_i(x) - \nabla V_i V_j(x)$.

(B4) For each $1 \leq i,j \leq d$, $(V^{-1})_j^i(\cdot) \in C_b^1(\mathbb{R}^d;\mathbb{R})$.

(B5) For each $1 \leq i \leq d$, $V_0^i(\cdot) \in C_b^1(\mathbb{R}^d;\mathbb{R})$.

Theorem 3. *Assume (B). Then there exists a random variable* $\hat{H}_T$ *such that* $\hat{H}_T$ *belongs to* $L^p(\Omega',\mathcal{F}',P')$ *for any* $p \geq 1$*, and*

$$E^{P'}[\varphi'(\hat{M}_T)] = E^{P'}[\varphi(\hat{M}_T)\hat{H}_T]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$*.*

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Identification of noncausal functions from the stochastic Fourier coefficients without the aid of a Brownian motion

Hideaki UEMURA (Aichi University of Education)

Let $f(t,\omega)$ be a random function on $[0,1]\times\Omega$ and $\{e_n(t)\}$ be a CONS in $L^2([0,1];\mathbb{C})$. The system $\{\int_0^1 f(t,\omega)\overline{e_n(t)}dW_t\}$ is called the stochastic Fourier coefficients (SFCs in abbr.) of $f(t,\omega)$. It is of course these stochastic integrals should be defined adequately. Let us consider whether $f(t,\omega)$ is identified from SFCs of $f(t,\omega)$.

Let $e_n(t)=\mathrm{e}^{2\pi int}$, $n\in\mathbb{Z}$. S.Ogawa [1], S.Ogawa and I [2,3] have studied this problem in the framework of the theory of the Skorokhod integral *with* the aid of a Brownian motion. Recently, however, S. Ogawa [4] obtained the affirmative answer *without* the aid of a Brownian motion from the stochastic Fourier transform if $f(t,\omega)$ is a nonnegative causal function. In this talk we will develop his method to the case where $f(t,\omega)$ is noncausal.

In this talk we assume the following conditions.

- $f(t,\omega)$ is differentiable with respect to t for almost all ω,
- $\displaystyle\int_0^1 f(t,\omega)dt\in L^2(\Omega,dP),\ f'(t,\omega)\Big(=\frac{\partial}{\partial t}f(t,\omega)\Big)\in L^2([0,1]\times\Omega,dtdP)$,
- $e_n(t)=\mathrm{e}^{2\pi int},\quad n\in\mathbb{Z}$.

We define the stochastic Fourier coefficients through the Ogawa integral. We use the symbol d_*W_t for Ogawa integral. We remark that $f(t,\omega)$ under our conditions is Ogawa integrable and satisfies

$$\int_0^1 f(t,\omega)d_*W_t=f(1,\omega)W_1-\int_0^1 W_tf'(t,\omega)dt. \tag{1}$$

We denote the SFC $\int_0^1 f(t,\omega)\overline{e_n(t)}d_*W_t$ by $\tilde{f}_n$.

Proposition 1. *$\{\tilde{f}_n,n\in\mathbb{Z}\}$ is uniformly bounded in $L^1(dP)$.*

Since it holds that

$$\lim_{N.M\to\infty}E\left[\sup_{0\leqq t\leqq 1}\left|\sum_{n\neq 0,|n|\leqq N}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)-\sum_{-4\pi^2n\neq 0,|n|\leqq M}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)\right|\right]=0,$$

Proposition 2. *There exists $S(t)(=S(t,\omega))\in C([0,1])$ a.s. such that*

$$\lim_{N\to\infty}E\left[\sup_{0\leqq t\leqq 1}\left|\sum_{n\neq 0,|n|\leqq N}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)-S(t)\right|\right]=0$$

We call $S(t)$ the $\{(-4\pi^2 n^2)^{-1}\}$-stochastic Fourier transform of $\{\hat{f}_n\}$.

From (1) and the integration by parts formula we have

$$\begin{aligned}S(t) = &-\frac{1}{2}\left(f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt\right)\left(\frac{1}{6} - t + t^2\right)\\ &-\left(\int_0^1 \left(\int_0^t W_s f'(s,\omega)ds\right)dt - \int_0^1 W_t f(t,\omega)dt\right)\left(\frac{1}{2} - t\right)\\ &-\left(\int_0^t \int_0^s W_u f'(u,\omega)duds - \int_0^1 \int_0^t \int_0^s W_u f'(u,\omega)dudsdt\right)\\ &+\left(\int_0^t W_s f(s,\omega)ds - \int_0^1 \int_0^t W_s f(s,\omega)dsdt\right)\end{aligned}$$

for all $t \in (0,1)$ almost surely. Since the right hand side above is differentiable in $t \in (0,1)$, so is $S(t)$ and we have

$$\begin{aligned}S'(t) = &-\frac{1}{2}\left(f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt\right)(-1+2t)\\ &+\left(\int_0^1 \left(\int_0^t W_s f'(s,\omega)ds\right)dt - \int_0^1 W_t f(t,\omega)dt\right)\\ &-\int_0^t W_u f'(u,\omega)du + W_t f(t,\omega)\end{aligned}$$

if $t \in (0,1)$. Thus it holds that for all fixed $s \in (0,1)$

$$\begin{aligned}&\limsup_{t\downarrow s} \frac{S'(t) - S'(s)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}}\\ &= \limsup_{t\downarrow s} \frac{W_t f(t,\omega) - W_s f(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega) \quad a.s. \qquad (2)\end{aligned}$$

We note that the set on which (2) fails depends on s. Set $\mathbb{S} \subset (0,1)$ be a countable dense subset, then we have

Theorem 1.

$$\limsup_{t\downarrow s} \frac{S'(t) - S'(s)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega)$$

for all $s \in \mathbb{S}$ almost surely. If $s \notin \mathbb{S}$, then $f(s,\omega) = \lim_{t\to s, t\in\mathbb{S}} f(t,\omega)$ holds.

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DIFFERENTIAL EQUATIONS DRIVEN BY ROUGH PATHS: AN APPROACH VIA FRACTIONAL CALCULUS

YU ITO

In this talk, I will consider differential equations driven by β-Hölder rough paths with $\beta \in (1/3, 1/2]$. First, on the basis of fractional calculus, I will introduce an integral of controlled paths along the rough paths (Eq. (2)). This can be regarded as an alternative approach to the integration introduced by M. Gubinelli [1]. Then, combining Eqs. (1) and (2), the solution of the differential equations will be defined by the same way introduced in [1]. Finally, as the main results of this talk, I will report the existence, uniqueness and continuity of the solution of the differential equations driven by geometric β-Hölder rough paths.

In the following, I will introduce some basic concepts which will be used in this talk.

Notation. Let V and W be finite-dimensional normed spaces. We use $L(V, W)$ to denote the set of all linear maps from V to W. Let T denote a positive constant and Δ_T denote the simplex $\{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. We use $\mathcal{C}_1^\lambda(V)$ to denote the space of all V-valued λ-Hölder continuous functions on the interval $[0, T]$. We use $\mathcal{C}_2(V)$ to denote the space of all V-valued continuous functions on Δ_T. Furthermore, for $\Psi \in \mathcal{C}_2(V)$ and $\mu > 0$, we set

$$\|\Psi\|_\mu := \sup_{0 \leq s < t \leq T} \frac{\|\Psi_{s,t}\|_V}{(t-s)^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) := \{\Psi \in \mathcal{C}_2(V) : \|\Psi\|_\mu < \infty\}.$$

Rough paths. Let $\beta \in (1/3, 1/2]$. We say that a continuous map $X = (X^1, X^2)$ from Δ_T to $\mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ is a β-Hölder rough path in $\mathbb{R}^d$ if X satisfies the following properties:

(1) for each $s, t, u \in [0, T]$ with $s \leq u \leq t$,

$$X^1_{s,t} = X^1_{s,u} + X^1_{u,t} \quad \text{and} \quad X^2_{s,t} = X^2_{s,u} + X^1_{s,u} \otimes X^1_{u,t} + X^2_{u,t};$$

(2) $X^1 \in \mathcal{C}_2^\beta(\mathbb{R}^d)$ and $X^2 \in \mathcal{C}_2^{2\beta}(\mathbb{R}^d \otimes \mathbb{R}^d)$.

The space of all β-Hölder rough paths in $\mathbb{R}^d$ is denoted by $\Omega_\beta(\mathbb{R}^d)$, which is a complete metric space whose distance is

$$d_\beta(X, \tilde{X}) := \|X^1 - \tilde{X}^1\|_\beta + \|X^2 - \tilde{X}^2\|_{2\beta}$$

for $X = (X^1, X^2), \tilde{X} = (\tilde{X}^1, \tilde{X}^2) \in \Omega_\beta(\mathbb{R}^d)$. Let $x \in \mathcal{C}_1^1(\mathbb{R}^d)$. We set

$$X^1_{s,t} := x_t - x_s \quad \text{and} \quad X^2_{s,t} := \int_s^t (x_u - x_s) \otimes dx_u$$

for $(s,t) \in \Delta_T$. Then we see that $X = (X^1, X^2)$ is a β-Hölder rough paths in $\mathbb{R}^d$. This is called smooth rough path or the step-2 signature of x. The elements in the closure of the set of all smooth rough paths with respect to the distance d_β are called geometric β-Hölder rough paths. The spaces of all smooth rough paths and geometric β-Hölder rough paths in $\mathbb{R}^d$ are denoted by $S\Omega_\beta(\mathbb{R}^d)$ and $G\Omega_\beta(\mathbb{R}^d)$, respectively.

Controlled paths. Let $X = (X^1, X^2) \in \Omega_\beta(\mathbb{R}^d)$. We say that a pair (Y, Y') is an $\mathbb{R}^e$-valued controlled path based on X if (Y, Y') satisfies the following properties:

(1) $Y \in \mathcal{C}_1^\beta(\mathbb{R}^e)$ and $Y' \in \mathcal{C}_1^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$;
(2) $R^Y \in \mathcal{C}_2^{2\beta}(\mathbb{R}^e)$, where $R^Y_{s,t} := Y_t - Y_s - Y'_s X^1_{s,t}$ for $(s,t) \in \Delta_T$.

The space of all $\mathbb{R}^e$-valued controlled paths based on X is denoted by $\mathcal{Q}_X^\beta(\mathbb{R}^e)$, which is a Banach space whose norm is

$$\|(Y, Y')\|_{X,\beta} := |Y_0| + |Y'_0| + \|R^Y\|_{2\beta} + \|\delta Y'\|_\beta, \quad (Y, Y') \in \mathcal{Q}_X^\beta(\mathbb{R}^e).$$

Here $\delta Y'_{s,t} := Y'_t - Y'_s$ for $(s,t) \in \Delta_T$. Let f be an $L(\mathbb{R}^d, \mathbb{R}^e)$-valued continuously Fréchet differentiable function on $\mathbb{R}^e$ whose derivative ∇f is Lipschitz continuous on $\mathbb{R}^e$. We set

$$Z_t := f(Y_t) \quad \text{and} \quad Z'_t := \nabla f(Y_t) Y'_t \tag{1}$$

for $t \in [0,T]$. Then we see that (Z, Z') belongs to $\mathcal{Q}_X^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$.

Integration of controlled paths via fractional calculus. Let $\Psi \in \mathcal{C}_2^\lambda(V)$ with $0 < \lambda \leq 1$. For $\alpha \in (0, \lambda)$, $s \in [0,T)$ and $t \in (0,T]$, we define $\mathcal{D}^\alpha_{s+}\Psi$ and $\mathcal{D}^\alpha_{t-}\Psi$ as $\mathcal{D}^\alpha_{s+}\Psi(s) := 0$,

$$\mathcal{D}^\alpha_{s+}\Psi(u) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{\Psi_{s,u}}{(u-s)^\alpha} + \alpha \int_s^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}} \, dv \right) \quad \text{for } u \in (s, T]$$

and $\mathcal{D}^\alpha_{t-}\Psi(t) := 0$,

$$\mathcal{D}^\alpha_{t-}\Psi(r) := \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \left(\frac{\Psi_{r,t}}{(t-r)^\alpha} + \alpha \int_r^t \frac{\Psi_{r,v}}{(v-r)^{\alpha+1}} \, dv \right) \quad \text{for } r \in [0, t),$$

where Γ is the Euler gamma function. If Ψ is of the form $\Psi_{s,t} = \psi_t - \psi_s$ for some $\psi \in \mathcal{C}_1^\lambda(V)$, then, from the definition, these functions coincide with the Weyl–Marchaud fractional derivatives of ψ of order α. Moreover, for $X = (X^1, X^2) \in \Omega_\beta(\mathbb{R}^d)$, $(Z, Z') \in \mathcal{Q}_X^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$ and $\gamma \in ((1-\beta)/2, \beta)$, an $\mathbb{R}^e$-valued function $I^\gamma(X, Z)$ on Δ_T is defined by

$$\begin{aligned} I^\gamma(X,Z)_{s,t} := Z_s X^1_{s,t} + Z'_s X^2_{s,t} &+ (-1)^{1-\gamma} \int_s^t \mathcal{D}^{1-\gamma}_{s+} R^Z(u) \mathcal{R}^{(1,\gamma)}_{t-} X(u) \, du \\ &+ (-1)^{1-2\gamma} \int_s^t \mathcal{D}^{1-2\gamma}_{s+} \delta Z'(u) \mathcal{R}^{(2,\gamma)}_{t-} X(u) \, du \end{aligned} \tag{2}$$

for $(s,t) \in \Delta_T$. Here $\delta Z'_{s,t} := Z'_t - Z'_s$, $\mathcal{R}^{(1,\gamma)}_{t-} X(u) := \mathcal{D}^\gamma_{t-} X^1(u)$ and

$$\mathcal{R}^{(2,\gamma)}_{t-} X(u) := \mathcal{D}^{2\gamma}_{t-} X^2(u) + \frac{(-1)^\gamma \gamma}{\Gamma(1-\gamma)} \int_u^t \frac{X^1_{u,v} \otimes \mathcal{R}^{(1,\gamma)}_{t-} X(v)}{(v-u)^{\gamma+1}} \, dv.$$

We refer to [2] for the details of $I^\gamma(X, Z)$ and the generalization for any $\beta \in (0, 1]$.

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Vortex solutions in Bose-Einstein Condensation

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This is a joint work with Anne de Bouard and Romain Poncet. The following nonlinear Schrödinger equation,

$$i\partial_t u = -\frac{\hbar^2}{2M}\Delta u + |x|^2 u + \lambda |u|^2 u, \quad t \geq 0, \quad x \in \mathbb{R}^d, \tag{1}$$

called Gross-Pitaevskii equation, was initially used as a model equation to describe magnetically trapped Bose gas. Bose gas is described by u, the wave function of the condensate, $\hbar$ is Planck's constant, M is the atomic mass of atoms in the condensate. The sign of λ is related to the sign of the atomic scattering length, which may be positive or negative.

We are interested in the influence of noise in the Gross-Pitaevskii equation (1) with a stochastic perturbation of the following form.

$$i\partial_t u = -\frac{\hbar^2}{2M}\Delta u + |x|^2 u + \lambda |u|^2 u + \varepsilon |x|^2 u \dot{\xi}(t), \quad t \geq 0, \quad x \in \mathbb{R}^d, \tag{2}$$

where $\dot{\xi}$ is a white noise in time with correlation function $\mathbb{E}(\dot{\xi}(t)\dot{\xi}(s)) = \delta_0(t-s)$. Here, δ_0 denotes the Dirac measure at the origin, and $\varepsilon > 0$. The product arising in the right hand side is interpreted in the Stratonovich sense, since the noise here naturally arises as the limit of processes with nonzero correlation length. We moreover assume that the noise is real valued. This model is proposed in [1], possibly with the addition of a damping term, to describe Bose-Einstein condensate wave function in an all-optical far-off resonance laser trap. In this model, the term $\dot{\xi}(t)$ represents the deviations of the laser intensity around its mean value. It is argued in [1] that some fluctuations of the laser intensity are observed in this case, and that one should take into account stochasticity in the dynamical behavior of the condensate in real situations.

From the point of view of nonlinear waves, the interesting phenomena is that the Gross-Pitaevskii equation, similarly to other nonlinear dispersive equations, supports various types of solitary wave solutions. In the two-dimensional setting in particular, there are vortex solutions of the form

$$u(t, r, \theta) = e^{-i\mu t} e^{im\theta} \psi(r), \tag{3}$$

where r, θ are polar coordinates, m is the vortex degree, μ is the chemical potential and $\psi(r)$ is the radial non-negative vortex profile. Stability of vortex solutions to diverse forms of nonlinear Schrödinger equations has drawn much attention in recent years.

In this talk, we introduce some results on the influence of random perturbations on the propagation of deterministic vortex solutions (3). Because of the presence of noise, a stable vortex would not persist in its form for all time. Thus an interesting question is how long the stable vortex can persist, compared to the noise strength ε. We theoretically prove that up to times of the order of ε^{-2}, the solution of (2) having the same symmetry properties as the vortex, decomposes into the sum of a randomly modulated vortex solution and a small remainder, and we derive the equations for the modulation parameter. In addition, we show that the first order of the remainder, as ε goes to zero, converges to a Gaussian process. Finally, some numerical simulations on the dynamics of the vortex solution in the presence of noise are presented.

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