Global Hölder Properties of the Density of the Solutions of SDEs with Singular Drift Coefficient

Gô Yûki
Ritsumeikan University and Japan Science and Technology Agency

Joint work with
Arturo Kohatsu-Higa*and Masafumi Hayashi†

1 Introduction

In this presentation, we will discuss the regularity of the density of the distribution of the solution of SDEs with singular drift coefficient. In the following, we always assume uniformly ellipticity for the diffusion coefficient.

Kusuoka and Stroock show that if coefficients of SDE are sufficiently smooth then there exists smooth density ([3]). Also, Bouleau and Hirsch show that if the coefficients are Lipschitz continuous then there exists density ([1]).

Recently, Fournier and Printems show that for one dimensional SDE if the diffusion coefficient σ is α-Hölder continuous with $\alpha > \frac{1}{2}$ and drift coefficient is at most linear growth then the solution of the SDE admits a density (see [2]). Also Bally announced that this result can be extended in multidimensional case with α-Hölder continuous σ, where $\alpha > 0$.

The above results do not guarantee Hölder continuity properties of the density. However, if the diffusion coefficient is deterministic and Fourier transform of the drift coefficient exists then we may show that Hölder continuity properties of the density.

2 Main Result

In this presentation, we consider following d-dimensional SDE:

$$X_t = x_0 + \int_0^t \sigma_j(s) dB_s^j + \int_0^t b(X_s) ds,$$

where $\sigma : [0, T] \to \mathbf{R}^d \times \mathbf{R}^d$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ are Borel measurable functions.

*Ritsumeikan University and Japan Science and Technology Agency.
†University of the Ryukyus and Japan Science and Technology Agency.

We assume following hypothesizes to coefficients:
$(A1)$: b is bounded.
$(A2)$: σ is uniformly elliptic and belongs to $L^2\left([0,T];\mathbf{R}^d\right)$.

We also assume that the drift coefficient b satisfies one of the following conditions.
$(A3)$: There exist constants $C_0>0$ and $\alpha>0$ such that

$$|\mathscr{F}b(\theta)| = \left|\int_{\mathbf{R}^d} e^{-i\theta\cdot x} b(x)dx\right| \le \frac{C_0}{(1+|\theta|)^k}$$

holds for $k := d-1+\alpha$.

$(A4)$: There exist constants $C_0>0$ and $\alpha>0$ such that

$$|\mathscr{F}b(\theta)| \le C_0 \prod_{l=1}^{d} \frac{1}{(1+|\theta_l|)^k}$$

holds for $k := 1-\frac{1}{d}+\alpha$.

Our approach is based on Fourier transform method. To prove the Hölder continuity of the density, we use following classical result.

Lemma 2.1. *Let X be a $\mathbf{R}^d$ valued random variable and φ be its characteristic function. If there exits $\eta \in (0,1)$ such that*

$$\int_{\mathbf{R}^d} |\theta|^\eta |\varphi(\theta)| d\theta < +\infty$$

then the density function of the law of X exists and is γ-Hölder continuous for any $0<\gamma<\eta$.

Under these assumptions and by using Lemma 2.1., we have following result.

Theorem 2.1. *Let $t \in (0,T]$. Assume that $(A1)$, $(A2)$ and $(A3)$ or $(A4)$ hold and there exist positive constants C, β and $\delta \in (0,t)$ such that*

$$\left|\mathbf{E}_Q\left[\exp\left(i\theta\cdot\int_s^t \sigma_j(u)dB_u^j\right)\right]\right| \le \exp(-C|\theta|^2(t-s)^\beta)$$

for any $s\in[t-\delta,t]$. Then the density function of the law of X_t exists and is λ-Hölder continuous for any $\lambda \in \left(0,(\alpha+\frac{2}{\beta}-2)\wedge 1\right)$.

References

[1] N. Bouleau and F. Hirsch, propriétés d'absolue continuité dans les espaces de Dirichlet et applications aux équations différentielles stochastiques, Séminarie de probabilités XX, Lecture Notes in Math. 1204(1986), 131-161.

[2] N. Fournier and J. Printems, Absolute continuity for some one dimensional processes, Bernoulli, 16(2), 2010, 343-360.

[3] S. Kusuoka and D. W. Stroock, Applications of the Malliavin calculus, Part I, Stochastic Analysis (Katata/Kyoto, 1982), North-Holland Math. Library, 271-306.