Asymptotics of quantum walks on the line

Graduate School of Mathematics, Nagoya University

Tatsuya TATE*

The notion of quantum walks, often called discrete time quantum random walks, was introduced by Aharonov-Davidovich-Zagury ([ADZ]) in 1993 as a quantum analogue of the classical random walks, and re-discovered in the area of computer science. In particular, Ambainis-Kempe-Rivosh ([AKR]) utilized two-dimensional quantum walks to improve Grover's quantum search algorithm. In the talk, various local asymptotic formulas of transition probabilities of quantum walks on the one-dimensional integer lattice, obtained in [ST], will be given. In the present article, we just mention one of the formulas, which is a limit formula of a large deviation type. To be precise, let us give a definition of quantum walks on the one-dimensional integer lattice. The quantum walks we consider in the talk is defined by a (special) unitary matrix,

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1,$$

and its decomposition,

$$A = P + Q, \quad P = \begin{pmatrix} a & 0 \\ -\bar{b} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & b \\ 0 & \bar{a} \end{pmatrix}.$$

Let $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ be the Hilbert space of square summable functions on $\mathbb{Z}$ with values in $\mathbb{C}^2$ whose inner product is given by

$$\langle f, g \rangle = \sum_{x \in \mathbb{Z}} \langle f(x), g(x) \rangle_{\mathbb{C}^2}, \quad f, g \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ denotes the standard inner product on $\mathbb{C}^2$. For any $u \in \mathbb{C}^2$ and $x \in \mathbb{Z}$, define $\delta_x \otimes u \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$(\delta_x \otimes u)(y) = \begin{cases} u & (y = x), \\ 0 & (y \neq x). \end{cases}$$

Then, the vectors, $\delta_x \otimes \mathbf{e}_i$ $(i = 1, 2,\ x \in \mathbb{Z})$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard orthonormal basis in $\mathbb{C}^2$, form an orthonormal basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. The unitary evolution, U, of the quantum walks on $\mathbb{Z}$ is a unitary operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined as

$$U = P\tau^{-1} + Q\tau,$$

where τ is the shift operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined by $\tau(\delta_x \otimes u) = \delta_{x+1} \otimes u$. The operator U is indeed a unitary operator, and hence the function

$$p_n(\varphi; x) = \|U^n(\delta_0 \otimes \varphi)(x)\|^2_{\mathbb{C}^2}, \quad x \in \mathbb{Z}$$

defines a probability distribution on $\mathbb{Z}$ supported on $[-n, n]$ for any unit vector φ in $\mathbb{C}^2$ and positive integer n, which we call the transition probability of the quantum walk U. The behavior of $p_n(\varphi; x)$ as $n \to \infty$ is one of main topics in the study of quantum walks. Indeed, as the following Figure 1† shows, it is drastically different from the behavior of transition probabilities of classical random walks. In Figure 1, the 'wall' is located at $x/n \sim \pm|a|$, where a is a component of the given unitary matrix A. The behavior of $p_n(\varphi; x)$ heavily depends on the 'normalized' position x/n according as

*E-mail: tate@math.nagoya-u.ac.jp

†Figure 1 is due to Dr. Takuya Machida in Meiji University.

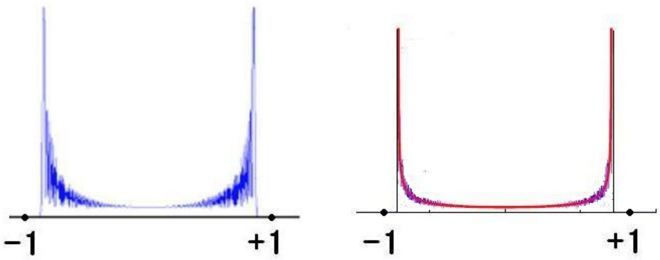

Figure 1: Probability disctribution and its weak-limit distribution

(1) x/n is inside the interval, $(-|a|, |a|)$,

(2) x/n stays around the 'wall', say, $x/n \sim \pm|a|$, or

(3) x/n is outside the interval, say, $|x/n| > |a|$.

Our analysis in [ST] gives precise asymptotic formulas in each region (1) – (3). For instance, a corollary to our results is stated as follows.

Corollary *Let $\xi \in \mathbb{R}$ satisfy $|a| < |\xi| < 1$. Suppose that a sequence of integers, $\{x_n\}$, satisfies*

$$x_n = n\xi + O(1) \quad (n \to \infty).$$

If $p_n(\varphi; x_n) \neq 0$ for every sufficiently large n, we have the following limit formula of a large deviation type,

$$\lim_{n\to\infty} \frac{1}{n} p_n(\varphi; x_n) = -H_Q(\xi),$$

where the function $H_Q(\xi)$ is given by

$$H_Q(\xi) = 2|\xi| \log\left(|b||\xi| + \sqrt{\xi^2 - |a|^2}\right) - 2\log\left(|b| + \sqrt{\xi^2 - |a|^2}\right) + (1 - |\xi|)\log(1 - \xi^2) - 2|\xi|\log|a|.$$

In the talk, after an explanation of backgrounds, properties and known results, such as a weak limit formula due to Konno ([K]), on the quantum walks on $\mathbb{Z}$ comparing with classical random walks, our main results on the asymptotic formulas of $p_n(\varphi; x)$ are introduced. According to our results, the asymptotic behavior of $p_n(\varphi; x)$ has indeed a quantum mechanical nature. The resemblance of the asymptotic behavior of $p_n(\varphi; x)$ and that of the Hermite functions will be pointed out by introducing the Plancerel-Rotach formula on asymptotic behavior of the Hermite functions.

References

[ADZ] Y. Aharonov, L. Davidovich and N. Zagury, Quantum random walks, *Phys. Rev. A*, **48**, no. 2 (1993), 1687–1690.

[AKR] A. Ambainis, J. Kempe and A. Rivosh, Coins make quantum walks faster, Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 1099?1108, ACM, New York, 2005.

[K] N. Konno, A new type of limit theorems for the one-dimensional quantum random walk, J. Math. Soc. Japan, vol. **57** (2005), 1179–1195.

[ST] T. Sunada and T. Tata, Asymptotic behavior of quantum walks on the line, J. Funct. Anal. **262** (2012), 2608–2645.