A Clark-Ocone type formula under change of measure for Lévy processes

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The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives that turns to be central in the application to mathematical finance. In this talk, we introduce a Clark-Ocone type formula under change of measure for Lévy processes with L^2-Lévy measure ([5, 6]). As an application of the theorem, we are also preparing a paper concerning the local risk minimization problem ([1]).

Throughout this talk, we consider Malliavin calculus for Lévy processes, based on, [4] and [2]. Let $X = \{X_t; t \in [0,T]\}$ be a centered square integrable Lévy process with representation

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmented filtration generated by X and σ is a constant number. Furthermore, we assume that $\{W_t; t \in [0,T]\}$ is a standard Brownian motion and that N is a Poisson random measure independent of W defined by

$$N(A,t) = \sum_{s\le t} \mathbf{1}_A(\Delta X_s), A \in \mathcal{B}(\mathbb{R}_0), \Delta X_s := X_s - X_{s-},$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. In addition, we will denote by $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ the compensated Poisson random measure, where $dt\nu(dz) = \lambda(dt)\nu(dz)$ is the compensator of N, $\nu(\cdot)$ the Lévy measure of X and λ the Lebesgue measure on $\mathbb{R}$. Since X is square integrable, the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Now we consider a finite measure q defined on $[0,T] \times \mathbb{R}$ by

$$q(E) = \sigma^2 \int_{E(0)} dt + \int_{E'} z^2\nu(dz)dt, \quad E \in \mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0) = \{t \in [0,T]; (t,0) \in E\}$ and $E' = E - E(0)$, and a random measure Q on $[0,T]\times\mathbb{R}$ by

$$Q(E) = \sigma \int_{E(0)} dW(t) + \int_{E'} z\tilde{N}(dt,dz), \quad E \in \mathcal{B}([0,T]\times\mathbb{R}).$$

Let $L^2_{T,q,n}(\mathbb{R})$ denote a set of product measurable, deterministic functions $f : ([0,T]\times\mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|f\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |f((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n \in \mathbb{N}$ and $f_n \in L^2_{T,q,n}(\mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q can be defined as

$$I_n(f_n) := \int_{([0,T]\times\mathbb{R})^n} f_n((t_1,z_1),\cdots,(t_n,z_n)) Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

In this setting, we introduce the following chaos expansion (see Theorem 2 in [3], Section 2 of [4]).

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Proposition 1 *Any $\mathcal{F}$-measurable square integrable random variable F has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \mathbb{P}-\text{a.s.}$$

with functions $f_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2_{T,q,n}}.$$

We next define the follows:

Definition 1 1. *Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying*

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2_{T,q,n}} < \infty.$$

2. *Let $F \in \mathbb{D}^{1,2}(\mathbb{R})$. Then the Malliavin derivative $DF : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}(\mathbb{R})$ is a stochastic process defined by*

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t, z), \cdot)), \quad \text{valid for } q-\text{a.e. } (t, z) \in [0, T] \times \mathbb{R}, \mathbb{P} - a.s.$$

3. *Let $\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s, x)|^2 q(ds, dx)\right] < \infty,$$

$G(s, x) \in \mathbb{D}^{1,2}(\mathbb{R}), q-\text{a.e. } (s, x) \in [0, T] \times \mathbb{R}$ *and*

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s, x)|^2 q(ds, dx) q(dt, dz)\right] < \infty.$$

4. *Let $\mathbb{L}^{1,2}_0(\mathbb{R})$ denote the space of measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0, T] \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]} |G(s)|^2 ds\right] < \infty,$$

$G(s) \in \mathbb{D}^{1,2}(\mathbb{R}),\ s \in [0, T], \text{a.e.}$ *and*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]} |D_{t,z}G(s)|^2 ds q(dt, dz)\right] < \infty.$$

5. *Let $\tilde{\mathbb{L}}^{1,2}_1(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0} |G(s, x)|^2 \nu(dx) ds\right] < \infty,$$

$$\mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G(s, x)| \nu(dx) ds\right)^2\right] < \infty,$$

$G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s,x) \in [0,T] \times \mathbb{R}_0$, a.e. ,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\left(\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G(s,x)|\nu(dx)ds\right)^2 q(dt,dz)\right] < \infty$$

and

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G(s,x)|^2\nu(dx)dsq(dt,dz)\right] < \infty.$$

Now, we assume the following.

Assumption 1 *Let $\theta(s,x) < 1, s \in [0,T], x \in \mathbb{R}_0$ and $u(s), s \in [0,T]$, be predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0}\{|\log(1-\theta(s,x))| + \theta^2(s,x)\}\nu(dx)ds < \infty, \text{ a.s.},$$
$$\int_0^T u^2(s)ds < \infty, \text{ a.s.}$$

Moreover we denote

$$Z(t) := \exp\left(-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u(s)^2 ds + \int_0^t\int_{\mathbb{R}_0}\log(1-\theta(s,x))\tilde{N}(ds,dx) + \int_0^t\int_{\mathbb{R}_0}(\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds\right), t \in [0,T].$$

Define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q}(\omega) = Z(\omega,T)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(s)ds + \int_0^T\int_{\mathbb{R}_0}\{(1-\theta(s,x))\log(1-\theta(s,x)) + \theta(s,x)\}\nu(dx)ds\right)\right] < \infty.$$

Furthermore we denote

$$\tilde{N}_{\mathbb{Q}}(dt,dx) := \theta(t,x)\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_{\mathbb{Q}}(t) := u(t)dt + dW(t).$$

Second, we assume the following.

Assumption 2 *We denote*

$$\tilde{H}(t,z) := \exp\left(-\int_0^T zD_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T (zD_{t,z}u(s))^2 ds + \int_0^T\int_{\mathbb{R}_0}\left[zD_{t,z}\theta(s,x) + \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x))\right]\nu(dx)ds + \int_0^T\int_{\mathbb{R}_0}\tilde{N}_{\mathbb{Q}}(ds,dx)\right),$$

and

$$K(t) := \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) + \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds,dx).$$

Furthermore, assume that the following:

1. *$F, Z(T) \in \mathbb{D}^{1,2}(\mathbb{R})$, with $(F + Z(T))^2 \in L^2(\mathbb{P})$,*

$$(F + Z(T))(D_{t,z}F + D_{t,z}Z(T)), z(D_{t,z}F + D_{t,z}Z(T))^2 \in L^2(q \times \mathbb{P}),$$

2. *$Z(T)D_{t,0} \log Z(T) \in L^2(\lambda \times \mathbb{P}), Z(T)(e^{zD_{t,z} \log Z(T)} - 1) \in L^2(\nu(dz)dtd\mathbb{P})$,*
3. *$u(s)D_{t,0}u(s) \in L^2(\lambda \times \mathbb{P}), 2u(s)D_{t,z}u(s) + z(D_{t,z}u(s))^2 \in L^2(z^2\nu(dz)dtd\mathbb{P})$, s-a.e.*
4. *$\log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right) \in L^2(\nu(dz)dtd\mathbb{P}), \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} \in L^2(\lambda \times \mathbb{P}), (s, x)$ -a.e.*
5. *$u, x^{-1} \log(1 - \theta(s, x)) \in \mathbb{L}^{1,2}(\mathbb{R})$,*
6. *$u(s)^2 \in \mathbb{L}_0^{1,2}$ and $\theta(s, x), \log(1 - \theta(s, x)) \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$,*
7. *and $F\tilde{H}(t, z), \tilde{H}(t, z)D_{t,z}F \in L^1(\mathbb{Q}), (t, z)$ -a.e.*

We next introduce a Clark-Ocone type formula under change of measure for Lévy processes.

Theorem 1 *Under Assumption 1 and Assumption 2, we have*

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{Q}}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t, z) - 1) + z\tilde{H}(t, z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt, dz). \end{aligned}$$

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