Pfaffian expressions for correlation functions of zeros of a Gaussian power series

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This is **a joint work with Tomoyuki Shirai (Kyushu University)**.

The zero distributions for Gaussian analytic functions have been studied for many years. Kac [1] gives an explicit expression for the probability density function of real zeros of a random polynomial $p_n(x) = \sum_{k=0}^{n} a_k x^k$, where a_k are i.i.d. real standard Gaussian random variables. Peres and Virág [2] study a random power series $f_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \zeta_k z^k$, where ζ_k are i.i.d. complex standard Gaussian random variables, and show that the zero distribution of $f_{\mathbb{C}}$ forms a determinantal point process associated with the Bergman kernel $K(z,w) = \frac{1}{(1-z\bar{w})^2}$.

We here consider a random power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where a_k are i.i.d. real standard Gaussian random variables. The random function f is a limiting version of Kac polynomial p_n and a real version of $f_{\mathbb{C}}$. From the Borel-Cantelli lemma, we see that the radius of convergence of f is almost surely 1. Furthermore, the restriction $\{f(t)\}_{t \in I}$ to the interval $I = (-1, +1)$ becomes a Gaussian process with covariance $\mathbb{E}[f(s)f(t)] = \frac{1}{1-st}$.

Our main results state that the zero distribution of f forms a Pfaffian point process. Recall the definition of the Pfaffian. For a $2n \times 2n$ skew-symmetric matrix $B = (b_{ij})_{1 \le i,j \le 2n}$, the Pfaffian of B is

$$\mathrm{Pf}\, B = \sum_{\sigma} \epsilon(\sigma) b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)}$$

summed over all permutations σ of $1, 2, \dots, 2n$ satisfying $\sigma(2i-1) < \sigma(2i)$ $(i = 1, 2, \dots, n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. Here $\epsilon(\sigma)$ is the signature of σ.

Theorem 1. *Let $\rho_n^{\mathrm{r}}(t_1, \dots, t_n)$ be the correlation function for real zeros of f. For $t_1, t_2, \dots, t_n \in I$, we have*

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) = \pi^{-n}\, \mathrm{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}.$$

Here each $\mathbb{K}(s,t)$ $(s, t \in I)$ is a 2×2 matrix and $\mathrm{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}$ is the Pfaffian of the $2n \times 2n$ skew-symmetric matrix $(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}$. The matrix kernel $\mathbb{K}(s,t)$ is defined as follows:

$$\mathbb{K}(s,t) = \begin{pmatrix} \mathbb{K}_{11}(s,t) & \mathbb{K}_{12}(s,t) \\ \mathbb{K}_{21}(s,t) & \mathbb{K}_{22}(s,t) \end{pmatrix}$$

and

$$\mathbb{K}_{11}(s,t) = \frac{s-t}{\sqrt{(1-s^2)(1-t^2)(1-st)^2}}, \qquad \mathbb{K}_{12}(s,t) = \sqrt{\frac{1-t^2}{1-s^2}}\frac{1}{1-st},$$
$$\mathbb{K}_{21}(s,t) = -\sqrt{\frac{1-s^2}{1-t^2}}\frac{1}{1-st}, \qquad \mathbb{K}_{22}(s,t) = \operatorname{sgn}(t-s)\arcsin\frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}.$$

Here $\operatorname{sgn} t = +1$ *for* $t > 0$*;* $\operatorname{sgn} t = -1$ *for* $t < 0$*; and* $\operatorname{sgn} 0 = 0$.

Theorem 2. *Let* $\rho_n^{\mathrm{c}}(z_1, \dots, z_n)$ *be the correlation function for complex zeros of* f*. For complex numbers* $z_1, \dots, z_n$ *satisfying* $|z_i| < 1$ *and* $\Im(z_i) > 0$*, we have*

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n}\prod_{j=1}^{n}\frac{1}{|1-z_j^2|} \cdot \operatorname{Pf}(\mathbb{K}^{\mathrm{c}}(z_i, z_j))_{1\le i,j\le n}$$

with

$$\mathbb{K}^{\mathrm{c}}(z,w) = \begin{pmatrix} \frac{z-w}{1-zw} & \frac{z-\bar{w}}{1-z\bar{w}} \\ \frac{\bar{z}-w}{1-\bar{z}w} & \frac{\bar{z}-\bar{w}}{1-\bar{z}\bar{w}} \end{pmatrix}$$

As corollaries of our proof of Theorem 1, we obtain the following Pfaffian expressions for absolute value moments and sign moments.

Theorem 3. *For distinct* $t_1, t_2, \dots, t_n \in I$*,*

$$\mathbb{E}[|f(t_1)f(t_2)\cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2}(\det\Sigma)^{-\frac{1}{2}}\operatorname{Pf}(\mathbb{K}(t_i, t_j))_{1\le i,j\le n},$$

with $\Sigma = \left(\frac{1}{1-t_it_j}\right)_{1\le i,j\le n}$.

Theorem 4. *For distinct* $t_1, t_2, \dots, t_{2n} \in I$*,*

$$\mathbb{E}[\operatorname{sgn} f(t_1)\operatorname{sgn} f(t_2)\cdots\operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^{n}\prod_{1\le i<j\le 2n}\operatorname{sgn}(t_j - t_i)\cdot\operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1\le i,j\le 2n}.$$

References

[1] M. Kac, On the average number of real roots of a random algebraic equation, Bull. Amer. Math. Soc. **49** (1943), 314–320.

[2] Y. Peres and B. Virág, Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process, Acta Math. **194** (2005), 1–35.