Integrals along rough paths via fractional calculus

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In [2], Hu and Nualart introduced an alternative approach to the rough path analysis, and it was based on the fractional calculus. This approach has produced the integral along β-Hölder continuous functions of order $\beta \in (1/3, 1/2)$ by combining the ideas of the rough path analysis and the techniques of the integration by parts formula in terms of the fractional derivatives introduced by Zähle ([3]). This integral has given a new tool to study the differential equations driven by β-Hölder continuous functions of order $\beta \in (1/3, 1/2)$; for example a study on the stochastic differential equations driven by the fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2)$ is found in [1].

One of the interesting points of this approach is that the integral is given by way to the usual Lebesgue integrals based on the fractional derivatives, and the definition does not require any approximation arguments, unlike the integrals in the context of the rough path analysis, namely the rough integrals. It is expected that the further development of this approach would provide a sophisticated access to the fundamental theory of the rough path analysis. Therefore, we arrive at the natural question to consider next, that is whether this approach is valid for any $\beta \in (0, 1]$; in particular, it drives us to the following question.

Question. Can the rough integrals be expressed as usual Lesbesgue integrals based on the fractional derivatives for any $\beta \in (0, 1]$?

In this talk, we will give an affirmative answer to this question, and produce the integral along β-Hölder rough paths based on the fractional derivatives for any $\beta \in (0, 1]$. It is a generalization of the preceding study of [2] in the following sense. This integral is a natural generalization of the Riemann–Stieltjes integral along smooth curves, and a continuous functional with respect to the β-Hölder topology under suitable conditions on the integrand. Consequently, we will obtain the following answer to the above question.

Answer. The first level path of the rough integrals along the geometric β-Hölder rough paths can be expressed as usual Lesbesgue integrals based on the fractional derivatives for any $\beta \in (0, 1]$.

References

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