Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians*

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Let $-L+V_\lambda$ be a spatially cut-off $P(\phi)_2$-Hamiltonian, where $\lambda=1/\hbar$ is a large positive parameter. The operator $-L$ is the free Hamiltonian, that is the second quantization operator of $\sqrt{m^2-\Delta}$, where m is a positive number. The potential function V_λ is given by a Wick polynomial

$$V_\lambda(w)=\lambda\int_{\mathbb{R}}:P\left(\frac{w(x)}{\sqrt{\lambda}}\right):g(x)dx, \tag{1}$$

where g is a smooth cut-off function and $P(x)=\sum_{k=1}^{2M}a_kx^k$ is a polynomial bounded from below. Formally, $-L_A+V_\lambda$ is unitarily equivalent to the infinite dimensional Schrödinger operator:

$$-\Delta_{L^2(\mathbb{R})}+\lambda U(w/\sqrt{\lambda})-\frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2}\quad \text{on } L^2(L^2(\mathbb{R}),dw) \tag{2}$$

where dw is an infinite dimensional Lebesgue measure. The function U is a potential function such that

$$U(w)=\frac{1}{4}\int_{\mathbb{R}}w'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}w(x)^2+:P(w(x)):g(x)\right)dx$$

and $\Delta_{L^2(\mathbb{R})}$ denotes the "Laplacian" on $L^2(\mathbb{R},dx)$. Hence, by the analogy of Schrödinger operators in finite dimensions, it is natural to expect that asymptotic behavior of lowlying eigenvalues of the spatially cut-off $P(\phi)_2$-Hamiltonian in the semiclassical limit $\lambda\to\infty$ is related with the global minimum points of U. In view of this, we consider the following assumptions.

Assumption 1. Let P be the polynomial in (1) and U be the function on H^1 which is given by

$$U(h)=\frac{1}{4}\int_{\mathbb{R}}h'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}h(x)^2+P(h(x))g(x)\right)dx \qquad \text{for } h\in H^1. \tag{3}$$

(A1) The function U is non-negative and the zero point set

$$\mathcal{Z}:=\{h\in H^1\mid U(h)=0\}=\{h_1,\dots,h_n\} \tag{4}$$

is a finite set.
(A2) For all $1\le i\le n$, the Hessian $\nabla^2U(h_i)$ is non-degenerate. That is, there exists $\delta_i>0$ for each i such that

$$\begin{aligned}\nabla^2U(h_i)(h,h)&:=\frac{1}{2}\int_{\mathbb{R}}h'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{2}h(x)^2+P''(h_i(x))g(x)h(x)^2\right)dx\\ &\ge\delta_i\|h\|^2_{L^2(\mathbb{R})}\qquad \text{for all } h\in H^1(\mathbb{R}).\end{aligned} \tag{5}$$

(A3) For all x, $P(x)=P(-x)$ and $\mathcal{Z}=\{h_0,-h_0\}$, where $h_0\neq 0$.

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Let $E_1(\lambda)$ be the lowest eigenvalue of $-L+V_\lambda$. The first main result is as follows.

Theorem 2. Assume that (A1) and (A2) hold. Let $E_1(\lambda)=\inf\sigma(-L+V_\lambda)$. Then

$$\lim_{\lambda\to\infty} E_1(\lambda)=\min_{1\le i\le n} E_i, \tag{6}$$

where

$$E_i=\inf\sigma(-L+Q_{v_i}) \tag{7}$$

and Q_{v_i} is given by

$$Q_{v_i}(w)=\int_{\mathbb{R}} :w(x)^2: v_i(x)dx, \quad v_i(x)=\frac{1}{2}P''(h_i(x))g(x). \tag{8}$$

Remark 3. In the case of finite dimensional Schrödinger operators, there exist eigenvalues near the approximate eigenvalues E_i when λ is large. In Theorem 2, if $E_i<m+\min_{1\le i\le n}E_i$, then the same results hold by the result of Hoegh-Krohn and Simon. However, if it is not the case, it is not clear and they may be embedded eigenvalues in the essential spectrum. Under the assumptions in Theorem 5, $E_2(\lambda)$ is an eigenvalue for large λ.

Let

$$E_2(\lambda)=\inf\left\{\sigma(-L+V_\lambda)\setminus\{E_1(\lambda)\}\right\}.$$

We can prove that $E_2(\lambda)-E_1(\lambda)$ is exponentially small when U is a symmetric double well potential function. The exponential decay rate is given by the Agmon distance which is defined below.

Definition 4. Let $0<T<\infty$ and $h,k\in H^1(\mathbb{R})$. Let $AC_{T,h,k}(H^1(\mathbb{R}))$ be the all absolutely continuous paths $c:[0,T]\to H^1(\mathbb{R})$ satisfying $c(0)=h, c(T)=k$. Let U be the potential function in (3). Assume U is non-negative. We define the Agmon distance between h,k by

$$d_U^{Ag}(h,k)=\inf\left\{\ell_U(c)\mid c\in AC_{T,h,k}(H^1(\mathbb{R}))\right\}, \tag{9}$$

where

$$\ell_U(c)=\int_0^T\sqrt{U(c(t))}\|c'(t)\|_{L^2}dt. \tag{10}$$

The following estimate is the second main result.

Theorem 5. Assume that U satisfies (A1),(A2),(A3). Then it holds that

$$\limsup_{\lambda\to\infty}\frac{\log\left(E_2(\lambda)-E_1(\lambda)\right)}{\lambda}\le -d_U^{Ag}(h_0,-h_0). \tag{11}$$