2012年度 確率解析とその周辺

2012年10月24日(水)–26日(金)

名古屋大学多元数理科学研究科 A-207講義室

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以下の要領で、本年度の「確率解析とその周辺」を開催いたします。ふるってご参加ください。

日 時：2012年10月24日(水)–26日(金)

場 所：名古屋大学 多元数理科学研究科　A-207講義室

世 話 人：
会田茂樹 (東北大学大学院理学研究科)
重川一郎 (京都大学大学院理学研究科)
河備浩司 (岡山大学自然科学研究科)
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2012年度 確率解析シンポジウム ホームページ
http://www.math.kyoto-u.ac.jp/probability/sympo/sa12/

プログラム

10月24日(水)

13:30–14:20　結城郷（立命館大学）　Go Yuki (Ritsumeikan)
Global Hölder Properties of the Density of the Solutions of SDEs with Singular Drift Coefficient (joint work with Arturo Kohatsu-Higa and Masafumi Hayashi)

14:30–15:20　松本詔（名古屋大学）　Sho Matsumoto (Nagoya)
Pfaffian expressions for correlation functions of zeros of a Gaussian power series

15:40–16:30　楠岡誠一郎 (東北大学)　Seiichiro Kusuoka (Tohoku)
Survey on the fourth moment theorem, Stein's method and related topic　(Part I)

16:40–17:30　伊藤悠（京都大学）　Yu Ito (Kyoto)
Integrals along rough paths via fractional calculus

10月25日(木)

10:00–10:50　矢野孝次（京都大学）　Kouji Yano (Kyoto)
Functional limit theorem for processes pieced together from excursions

11:00–11:50　佐久間紀佳（愛知教育大学）　Noriyoshi Sakuma (Aichi Uni. of Education)
Variance mixture and subordinator in free probability

— **Lunch Break** —

13:30–14:20　Jan Maas (Bonn)
Approximating rough stochastic PDEs

14:30–15:20　楠岡誠一郎(東北大学)　Seiichiro Kusuoka (Tohoku)
Survey on the fourth moment theorem, Stein's method and related topic (Part II)

15:40–16:30　楯辰哉（名古屋大学）　Tatsuya Tate (Nagoya)
Asymptotics of quantum walks on the line

16:40–17:30　重川一郎（京都大学）　Ichiro Shigekawa (Kyoto)
Exponential convergence of Markovian semigroups (joint work with Seiichiro Kusuoka)

10月26日(金)

9:30–10:20　道工勇（埼玉大学）　Isamu Doku (Saitama)
Historical superprocess related to random measure.

10:30–11:20　鈴木良一(慶応義塾大学)　Ryoichi Suzuki (Keio)
A Clark-Ocone type formula under change of measure for Lévy processes

11:40–12:30　会田茂樹(東北大学)　Shigeki Aida (Tohoku)
Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

Global Hölder Properties of the Density of the Solutions of SDEs with Singular Drift Coefficient

Gô Yûki
Ritsumeikan University and Japan Science and Technology Agency

Joint work with
Arturo Kohatsu-Higa*and Masafumi Hayashi†

1 Introduction

In this presentation, we will discuss the regularity of the density of the distribution of the solution of SDEs with singular drift coefficient. In the following, we always assume uniformly ellipticity for the diffusion coefficient.

Kusuoka and Stroock show that if coefficients of SDE are sufficiently smooth then there exists smooth density ([3]). Also, Bouleau and Hirsch show that if the coefficients are Lipschitz continuous then there exists density ([1]).

Recently, Fournier and Printems show that for one dimensional SDE if the diffusion coefficient σ is α-Hölder continuous with $\alpha > \frac{1}{2}$ and drift coefficient is at most linear growth then the solution of the SDE admits a density (see [2]). Also Bally announced that this result can be extended in multidimensional case with α-Hölder continuous σ, where $\alpha > 0$.

The above results do not guarantee Hölder continuity properties of the density. However, if the diffusion coefficient is deterministic and Fourier transform of the drift coefficient exists then we may show that Hölder continuity properties of the density.

2 Main Result

In this presentation, we consider following d-dimensional SDE:

$$X_t = x_0 + \int_0^t \sigma_j(s) dB_s^j + \int_0^t b(X_s) ds,$$

where $\sigma : [0,T] \to \mathbf{R}^d \times \mathbf{R}^d$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ are Borel measurable functions.

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We assume following hypothesizes to coefficients:
$(A1)$: b is bounded.
$(A2)$: σ is uniformly elliptic and belongs to $L^2\left([0,T];\mathbf{R}^d\right)$.

We also assume that the drift coefficient b satisfies one of the following conditions.
$(A3)$: There exist constants $C_0 > 0$ and $\alpha > 0$ such that

$$|\mathscr{F}b(\theta)| = \left|\int_{\mathbf{R}^d} e^{-i\theta\cdot x} b(x)dx\right| \leq \frac{C_0}{(1+|\theta|)^k}$$

holds for $k := d-1+\alpha$.

$(A4)$: There exist constants $C_0 > 0$ and $\alpha > 0$ such that

$$|\mathscr{F}b(\theta)| \leq C_0 \prod_{l=1}^{d} \frac{1}{(1+|\theta_l|)^k}$$

holds for $k := 1 - \frac{1}{d} + \alpha$.

Our approach is based on Fourier transform method. To prove the Hölder continuity of the density, we use following classical result.

Lemma 2.1. *Let X be a $\mathbf{R}^d$ valued random variable and φ be its characteristic function. If there exits $\eta \in (0,1)$ such that*

$$\int_{\mathbf{R}^d} |\theta|^\eta |\varphi(\theta)| d\theta < +\infty$$

then the density function of the law of X exists and is γ-Hölder continuous for any $0 < \gamma < \eta$.

Under these assumptions and by using Lemma 2.1., we have following result.

Theorem 2.1. *Let $t \in (0,T]$. Assume that $(A1)$, $(A2)$ and $(A3)$ or $(A4)$ hold and there exist positive constants C, β and $\delta \in (0,t)$ such that*

$$\left|\mathbf{E}_Q\left[\exp\left(i\theta\cdot\int_s^t \sigma_j(u)dB_u^j\right)\right]\right| \leq \exp(-C|\theta|^2(t-s)^\beta)$$

for any $s \in [t-\delta, t]$. Then the density function of the law of X_t exists and is λ-Hölder continuous for any $\lambda \in \left(0, (\alpha + \frac{2}{\beta} - 2) \wedge 1\right)$.

References

[1] N. Bouleau and F. Hirsch, propriétés d'absolue continuité dans les espaces de Dirichlet et applications aux équations différentielles stochastiques, Séminarie de probabilités XX, Lecture Notes in Math. 1204(1986), 131-161.

[2] N. Fournier and J. Printems, Absolute continuity for some one dimensional processes, Bernoulli, 16(2), 2010, 343-360.

[3] S. Kusuoka and D. W. Stroock, Applications of the Malliavin calculus, Part I, Stochastic Analysis (Katata/Kyoto, 1982), North-Holland Math. Library, 271-306.

Pfaffian expressions for correlation functions of zeros of a Gaussian power series

Sho Matsumoto (Nagoya University)

This is **a joint work with Tomoyuki Shirai (Kyushu University).**

The zero distributions for Gaussian analytic functions have been studied for many years. Kac [1] gives an explicit expression for the probability density function of real zeros of a random polynomial $p_n(x) = \sum_{k=0}^{n} a_k x^k$, where a_k are i.i.d. real standard Gaussian random variables. Peres and Virág [2] study a random power series $f_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \zeta_k z^k$, where ζ_k are i.i.d. complex standard Gaussian random variables, and show that the zero distribution of $f_{\mathbb{C}}$ forms a determinantal point process associated with the Bergman kernel $K(z,w) = \frac{1}{(1-z\bar{w})^2}$.

We here consider a random power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where a_k are i.i.d. real standard Gaussian random variables. The random function f is a limiting version of Kac polynomial p_n and a real version of $f_{\mathbb{C}}$. From the Borel-Cantelli lemma, we see that the radius of convergence of f is almost surely 1. Furthermore, the restriction $\{f(t)\}_{t \in I}$ to the interval $I = (-1, +1)$ becomes a Gaussian process with covariance $\mathbb{E}[f(s)f(t)] = \frac{1}{1-st}$.

Our main results state that the zero distribution of f forms a Pfaffian point process. Recall the definition of the Pfaffian. For a $2n \times 2n$ skew-symmetric matrix $B = (b_{ij})_{1 \le i,j \le 2n}$, the Pfaffian of B is

$$\operatorname{Pf} B = \sum_{\sigma} \epsilon(\sigma) b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)}$$

summed over all permutations σ of $1, 2, \dots, 2n$ satisfying $\sigma(2i-1) < \sigma(2i)$ $(i = 1, 2, \dots, n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. Here $\epsilon(\sigma)$ is the signature of σ.

Theorem 1. *Let $\rho_n^{\mathrm{r}}(t_1, \dots, t_n)$ be the correlation function for real zeros of f. For $t_1, t_2, \dots, t_n \in I$, we have*

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) = \pi^{-n} \operatorname{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}.$$

Here each $\mathbb{K}(s,t)$ $(s, t \in I)$ is a 2×2 matrix and $\operatorname{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}$ is the Pfaffian of the $2n \times 2n$ skew-symmetric matrix $(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}$. The matrix kernel $\mathbb{K}(s,t)$ is defined as follows:

$$\mathbb{K}(s,t) = \begin{pmatrix} \mathbb{K}_{11}(s,t) & \mathbb{K}_{12}(s,t) \\ \mathbb{K}_{21}(s,t) & \mathbb{K}_{22}(s,t) \end{pmatrix}$$

and

$$\mathbb{K}_{11}(s,t) = \frac{s-t}{\sqrt{(1-s^2)(1-t^2)(1-st)^2}}, \quad \mathbb{K}_{12}(s,t) = \sqrt{\frac{1-t^2}{1-s^2}}\frac{1}{1-st},$$
$$\mathbb{K}_{21}(s,t) = -\sqrt{\frac{1-s^2}{1-t^2}}\frac{1}{1-st}, \quad \mathbb{K}_{22}(s,t) = \operatorname{sgn}(t-s)\arcsin\frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}.$$

Here $\operatorname{sgn} t = +1$ *for* $t > 0$*;* $\operatorname{sgn} t = -1$ *for* $t < 0$*; and* $\operatorname{sgn} 0 = 0$.

Theorem 2. *Let* $\rho_n^{\mathrm{c}}(z_1, \dots, z_n)$ *be the correlation function for complex zeros of* f*. For complex numbers* $z_1, \dots, z_n$ *satisfying* $|z_i| < 1$ *and* $\Im(z_i) > 0$*, we have*

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n}\prod_{j=1}^{n}\frac{1}{|1-z_j^2|}\cdot \operatorname{Pf}(\mathbb{K}^{\mathrm{c}}(z_i, z_j))_{1\le i,j\le n}$$

with

$$\mathbb{K}^{\mathrm{c}}(z,w) = \begin{pmatrix} \frac{z-w}{1-zw} & \frac{z-\bar{w}}{1-z\bar{w}} \\ \frac{\bar{z}-w}{1-\bar{z}w} & \frac{\bar{z}-\bar{w}}{1-\bar{z}\bar{w}} \end{pmatrix}$$

As corollaries of our proof of Theorem 1, we obtain the following Pfaffian expressions for absolute value moments and sign moments.

Theorem 3. *For distinct* $t_1, t_2, \dots, t_n \in I$*,*

$$\mathbb{E}[|f(t_1)f(t_2)\cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2}(\det\Sigma)^{-\frac{1}{2}}\operatorname{Pf}(\mathbb{K}(t_i,t_j))_{1\le i,j\le n},$$

with $\Sigma = \left(\frac{1}{1-t_it_j}\right)_{1\le i,j\le n}$.

Theorem 4. *For distinct* $t_1, t_2, \dots, t_{2n} \in I$*,*

$$\mathbb{E}[\operatorname{sgn} f(t_1)\operatorname{sgn} f(t_2)\cdots\operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^{n}\prod_{1\le i<j\le 2n}\operatorname{sgn}(t_j-t_i)\cdot\operatorname{Pf}(\mathbb{K}_{22}(t_i,t_j))_{1\le i,j\le 2n}.$$

References

[1] M. Kac, On the average number of real roots of a random algebraic equation, Bull. Amer. Math. Soc. **49** (1943), 314–320.

[2] Y. Peres and B. Virág, Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process, Acta Math. **194** (2005), 1–35.

Survey on the fourth moment theorem, Stein's method and related topics

Seiichiro Kusuoka[1] (Tohoku University)

1 Introduction

The fourth moment theorem was originally introduced by Nualart and Peccati [11]. The theorem gives some equivalent conditions for a sequence of random variables belonging to a level of Wiener chaos to convergent to the standard normal distribution. The most surprising part of the theorem is that; if the variances of the sequence converge to 1, then the convergence to the standard normal distribution is equivalent to the convergence of the fourth moments of the sequence to 3. After that, Nualart and Ortiz-Latorre [10] gave another equivalent condition and made a clearer proof in their paper. Stimulated by Nualart and Ortiz-Latorre [10], Nourdin and Peccati [6] discovered a new method to estimate distances between the standard normal distribution and other distributions, and between the centered Gamma distributions and other distributions. The method is a combination of Stein's method and Malliavin calculus. Nourdin and Peccati's method enables us to prove a part of the fourth moment theorem in another way. Now applications and other versions of the fourth moment theorem and Stein's bound are considered.

In this talk, we review the fourth moment theorem and Stein's method mainly, give a short review of further results and related topics.

Now we give some useful information. A textbook [7] written by Nourdin and Peccati was published recently. This book covers from the elementary tools for this topic to the fourth moment theorem and the density estimates obtained by Stein's method. The latest results on this topic are found on the webpage:

http://www.iecn.u-nancy.fr/ nourdin/steinmalliavin.htm

Many of literatures (e.g. lecture notes, articles) are listed up on this webpage.

2 Preliminary on Wiener chaos

First we prepare the elementary things on Wiener chaos.

Let $(T, \mathscr{B})$ be a measurable space, μ a σ-finite measure on $(T, \mathscr{B})$ without atoms, and $H := L^2(T, \mathscr{B}, \mu)$. We introduce the isonormal Gaussian process with respect to H. Let $W = \{W(h);\ h \in H\}$ be a family of random variables on a complete probability space $(\Omega, \mathscr{F}, P)$.

Definition 2.1. *We call W is an isonormal Gaussian process (or Gaussian process on H) if the following conditions hold.*

(i) *W is a Gaussian family (or a Gaussian system), i.e. for $n \in \mathbb{N}$ and $h_1, h_2, \dots, h_n \in H$, the $\mathbb{R}^n$-valued random variable $(W(h_1), W(h_2), \dots, W(h_n))$ has an n-dimensional Gaussian distribution.*

(ii) *H is the Cameron-Martin space (or the reproducing kernel Hilbert space) of W, i.e.*

$$E[W(h)] = 0, \qquad h \in H, \tag{2.1}$$

$$E[W(g)W(h)] = (g, h)_H, \qquad g, h \in H. \tag{2.2}$$

Let $W(A) := W(\mathbb{I}_A)$ for $A \in \mathscr{B}$ and $\mu(A) < \infty$. Then, the law of $\mathbb{W}$ is also characterized by $\{W(A); A \in \mathscr{B}, \mu(A) < \infty\}$, since L^2-functions are approximated by simple functions (step functions, elementary functions). The following assertions follows immediately from Definition 2.1.

(i) $W(A)$ has the distribution $N(0, \mu(A))$ for $A \in \mathscr{B}$ such that $\mu(A) < \infty$.

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(ii) $W(A_1)$ and $W(A_2)$ are independent of each other for $A_1, A_2 \in \{A \in \mathscr{B}; \mu(A) < \infty\}$ such that $A_1 \cap A_2 = \emptyset$.

(iii) $A \mapsto W(A)$ is an $L^2(\Omega, \mathscr{F}, P)$-valued finitely additive measure on $(T, \mathscr{B})$.

Note that $A \mapsto W(A)$ is not σ-additive.

Now, we start with the construction of multiple stochastic integrals. Let $m \in \mathbb{N}$ and $\mathscr{B}_0 := \{A \in \mathscr{B}; \mu(A) < \infty\}$. We define the multiple stochastic integral $I_m(f)$ of $f \in L^2(T^m, \mathscr{B}^{\otimes m}, \mu^{\otimes m})$ as follows. Let $\mathcal{E}_m$ be the total set of the functions f such that

$$f(t_1, t_2, \dots, t_m) = \sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1, i_2, \dots, i_m} \mathbb{I}_{A_{i_1} \times A_{i_2} \times \cdots A_{i_m}}(t_1, t_2, \dots, t_m) \tag{2.3}$$

where $n \in \mathbb{N}$, $A_1, A_2, \dots, A_n$ are pairwise-disjoint sets in $\mathscr{B}_0$, and $a_{i_1, i_2, \dots, i_m} \in \mathbb{R}$ such that $a_{i_1, i_2, \dots, i_m} = 0$ if $i_k = i_l$ for some $k, l = 1, 2, \dots, n$. Note that $\mathcal{E}_m$ is a linear space. For f expressed as in (2.3) we define

$$I_m(f) := \sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1, i_2, \dots, i_m} W(A_{i_1}) W(A_{i_2}) \dots W(A_{i_m}).$$

For $f \in L^2(T^m, \mathscr{B}^{\otimes m}, \mu^{\otimes m})$, define the symmetrization $\tilde{f}$ of f by

$$\tilde{f}(t_1, t_2, \dots, t_m) = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} f(t_{\pi(1)}, t_{\pi(2)}, \dots, t_{\pi(m)})$$

where $\mathfrak{S}_m$ is the group of permutations of $\{1, 2, \dots, m\}$. Note that the mapping $f \mapsto \tilde{f}$ from $L^2(\mu^{\otimes m})$ to itself is linear and continuous for each $m \in \mathbb{N}$. We call f symmetric if $f = \tilde{f}$. Let $H^{\odot n} := \{f \in H^{\otimes n};\ f \text{ is symmetric}\}$, and $\|f\|_{H^{\odot n}} := \sqrt{n!}\|f\|_{H^{\otimes n}}$ for $f \in H^{\odot n}$. Then, following properties hold.

Proposition 2.2. (i) *I_m is a linear mapping form $\mathcal{E}_m$ to $L^2(\Omega, \mathscr{F}, P)$.*

(ii) *For $f \in \mathcal{E}_m$, $I_m(f) = I_m(\tilde{f})$.*

(iii) *For $f \in \mathcal{E}_m$ and $g \in \mathcal{E}_q$,*

$$E[I_m(f) I_q(g)] = \begin{cases} 0 & \text{if } \ m \neq q, \\ m!(\tilde{f}, \tilde{g})_{L^2(\mu^{\otimes m})} & \text{if } \ m = q. \end{cases}$$

By the property (iii)

$$E[I_m(f)^2] = m!\|\tilde{f}\|^2_{L^2(\mu^{\otimes m})} \leq m!\|f\|^2_{L^2(\mu^{\otimes m})}. \tag{2.4}$$

By (2.4) we have

$$E[I_m(f)^2] = \|f\|^2_{H^{\odot n}}, \quad f \in H^{\odot n}.$$

The following lemma holds.

Lemma 2.3. *$\mathcal{E}_m$ is dense in $L^2(\mu^{\otimes m})$.*

By (2.4) and Lemma 2.3 we can extend I_m to a bounded linear operator from $L^2(T^m, \mathscr{B}^{\otimes m}, \mu^{\otimes m})$ to $L^2(\Omega, \mathscr{F}, P)$. The extension of I_m also satisfies the properties in Proposition 2.2 again.

By using Hermite polynomial, we have the following theorem. The theorem is called the Wiener-Chaos expansion.

Theorem 2.4. *Assume that $\mathscr{F}$ is the σ-field generated by $W = \{W(h);\ h \in H\}$. Then, for any $F \in L^2(\Omega, \mathscr{F}, P)$, there exist symmetric functions $\{f_n \in H^{\odot n}; n = 0, 1, 2, \dots\}$ such that $f_0 = E[F]$ and*

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

The functions $\{f_n\}$ are uniquely determined by F.

3 Preliminary on H-derivative

Let $\mathcal{P}$ be the class of the random variables F such that; there exist $n \in \mathbb{N}$, a polynomial function f on $\mathbb{R}^n$, $h_1, h_2, \dots, h_n \in H$, and F is expressed by

$$F = f(W(h_1), W(h_2), \dots, W(h_n)). \tag{3.1}$$

Then, the following lemma holds.

Lemma 3.1. *$\mathcal{P}$ is dense in $L^p(\Omega, \mathscr{F}, P)$ for $p \in [1, \infty)$.*

We define the H-derivative operator D as follows.

Definition 3.2. *For $F \in \mathcal{P}$ expressed as in (3.1), define the H-valued random variable DF of F by*

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), W(h_2), \dots, W(h_n)) h_i.$$

We call DF by the H-derivative (or Malliavin's derivative) of F.

It is easy to see that D is linear on $\mathcal{P}$ and D maps $\mathcal{P}$ into $L^p(\Omega, \mathscr{F}, P)$ for $p \in [1, \infty)$. Moreover, the following lemma holds.

Lemma 3.3. *D is a closable operator on $L^p(\Omega, \mathscr{F}, P)$ for $p \in [1, \infty)$.*

By Lemmas 3.1 and 3.3, D can be extended to a closed (unbounded) linear operator on $L^p(\Omega, \mathscr{F}, P)$ for $p \in [1, \infty)$. We denote the extension by D again.

For $F \in \mathcal{P}$ and $p \in [1, \infty)$, define $\|F\|_{1,p}$ by

$$\|F\|_{1,p} := \left(E[|F|^p] + \|DF\|_H^p\right)^{1/p}.$$

It is easy to see that $\|\cdot\|_{1,p}$ satisfies the properties of norms, and we can consider the closure of $\mathcal{P}$ with respect to $\|\cdot\|_{1,p}$. We denote the closure by $\mathbb{D}^{1,p}$. Note that $\|\cdot\|_{1,p}$ is the operator norm of D in $L^p(\Omega, \mathscr{F}, P)$. The function space $\mathbb{D}^{1,p}$ is the Sobolev space associated with the H-derivative with index $1, p$. Similarly, we can define the Sobolev space $\mathbb{D}^{k,p}$ of higher orders.

Proposition 3.4. *Let $F \in \mathbb{D}^{1,2}$ such that $F = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n \in H^{\odot n}$. Then,*

$$(DF, h)_H = \sum_{n=1}^{\infty} n \int_T I_{n-1}(f_n(\cdot, t)) h(t) \mu(dt)$$

where $f_n(\cdot, t)$ is the function on T^{n-1} given by

$$[f_n(\cdot, t)](s_1, s, 2, \dots, s_n - 1) := f_n(s_1, s_2, \dots, s_{n-1}, t), \quad s_1, s_2, \dots, s_n \in T.$$

Hence,

$$E[\|DF\|_H^2] = \sum_{n=1}^{\infty} n n! \|f_n\|_{H^{\otimes n}}^2 = \sum_{n=1}^{\infty} n \|f_n\|_{H^{\odot n}}^2.$$

Now we define the operator δ as the follows.

Definition 3.5. *Let δ be the dual operator of $D : L^2(\Omega, \mathscr{F}, P) \to L^2(\Omega, \mathscr{F}, P; H)$.*

The operator δ is called the Skorohod integral. We remark that the Skorohod integral can be regarded as an extension of the stochastic integral (Itô integral).

Let L be the Ornstein-Uhlembeck operator on $L^2(\Omega, \mathscr{F}, P)$ associated with W. There are some ways to define the Ornstein-Uhlembeck operator. For example, in [9] the Ornstein-Uhlembeck operator is defined by using Wiener chaos expansion. On the other hand, in [13], first we define the Ornstein-Uhlembeck semigroup by using the explicit transition semigroup, and the Ornstein-Uhlembeck operator is defined by the generator of the Ornstein-Uhlembeck semigroup. We omit the precise definition of L here, and only remark that the domain of L includes $\mathcal{P}$ and L is characterized by

$$LF = \sum_{i,j=1}^{n} \partial_i \partial_j f(W(h_1), W(h_2), \ldots, W(h_n))(h_i, h_j)_H - \sum_{i=1}^{n} \partial_i f(W(h_1), W(h_2), \ldots, W(h_n))W(h_i) \tag{3.2}$$

where F is the random variable expressed as (3.1).

The following propositions hold.

Proposition 3.6. $\delta D = -L$.

Proposition 3.7. *For* $f \in H^{\otimes n}$, $LI_n(f) = -nI_n(f)$.

We use these facts in the proofs of the fourth moment theorem and the Stein's bound.

4 The fourth moment theorem

In this section, we give the version of the fourth moment theorem given by Nualart and Ortiz-Latorre [10].

First we define the contraction of functions. For $f \in L^2(T^p, \mathscr{B}^{\otimes p}, \mu^{\otimes p})$, $g \in L^2(T^q, \mathscr{B}^{\otimes q}, \mu^{\otimes q})$ and $r = 1, 2, \ldots, \min\{p, q\}$, we define $f \otimes g \in L^2(T^{p+q}, \mathscr{B}^{\otimes p+q}, \mu^{\otimes p+q})$ and $f \otimes_r g \in L^2(T^{p+q-2r}, \mathscr{B}^{\otimes p+q-2r}, \mu^{\otimes p+q-2r})$ by

$$\begin{aligned}
&(f \otimes g)(t_1, t_2, \ldots, t_{p+q}) \\
&= f(t_1, t_2, \ldots, t_p)g(t_{p+1}, t_{p+2}, \ldots, t_{p+q}), \\
&(f \otimes_r g)(t_1, t_2, \ldots, t_{p+q-2r}) \\
&= \int_{T^r} f(t_1, t_2, \ldots, t_{p-r}, s_1, s_2, \ldots, s_r)g(t_{p-r+1}, t_{p-r+2}, \ldots, t_{p+q-2r}, s_1, s_2, \ldots, s_r) \\
&\qquad \times \mu^{\otimes r}(ds_1, ds_2, \ldots, ds_r),
\end{aligned}$$

respectively. We call the operation $(f, g) \mapsto f \otimes_r g$ is called the contraction of f and g of order r. Since $f \otimes g$ can be regarded as $f \otimes_r g$ with $r = 0$, we define $f \otimes_0 g$ by $f \otimes g$.

The tensor product $f \otimes g$ and the contractions $f \otimes_r g$ are not always symmetric even if f and g are symmetric. We denote the symmetrizations of $f \otimes g$ and $f \otimes_r g$ by $f \widetilde{\otimes} g$ and $f \widetilde{\otimes}_r g$, respectively.

By using contraction we can calculate the product of two random variables in some levels of Wiener chaos as follows.

Proposition 4.1. *Let* $f \in L^2(\mu^{\otimes p})$ *be symmetric and* $g \in L^2(\mu)$. *Then,*

$$I_p(f)I_1(g) = I_{p+1}(f \otimes g) + pI_{p-1}(f \otimes_1 g). \tag{4.1}$$

The proposition 4.1 is extended as follows.

Proposition 4.2. *Let* $f \in L^2(\mu^{\otimes p})$ *and* $g \in L^2(\mu^{\otimes q})$ *are symmetric. Then,*

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{4.2}$$

Note that Proposition 4.2 gives the explicit information of the Wiener chaos expansion of the product of two random variables in some levels of Wiener chaos. The proposition is crucial to prove the fourth moment theorem. In the proof of Proposition 4.2, complicated calculation in combination is needed.

By using Proposition 4.2, we have the fourth moment theorem which is the version given by Nualart and Ortiz-Latorre [10] as follows.

Theorem 4.3. *(The fourth moment theorem) Consider a sequence $\{F_k = I_n(f_k)\}$ of square integrable random variables in the n-th Wiener chaos. Assume that*

$$\lim_{k\to\infty} E[F_k^2] = \lim_{k\to\infty} \|f_k\|^2_{H^{\odot n}} = 1. \tag{4.3}$$

Then, the following statements are equivalent.

(i) $\{F_k = I_n(f_k)\}$ *converges to the standard normal law in distribution as* $k \to \infty$.

(ii) $\lim_{k\to\infty} E[F_k^4] = 3$.

(iii) $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{H^{\otimes 2(n-l)}} = 0$ *for* $l = 1, 2, \ldots, n-1$.

(iv) $\|DF_k\|^2_H$ *converges to* n *in* $\mathbb{L}^2$ *as* $k \to \infty$.

Multidimensional case of the fourth moment theorem is considered in [12], and [10]. In [6] the fourth moment theorem with respect to the centered Gamma distribution is also obtained.

5 Stein's method and Application of Malliavin calculus

Charles Stein considered in order to estimate the reminder term of the central limit theorem (see [14]). He prepared the ordinary differential equation associated with the standard normal distribution satisfies, and obtained a bound of the reminder term by using the solution to the equation. The equation is called Stein's equation, and the method to obtain the bound is called Stein's method. The large deviation principle is also well-known as a method to obtain the convergence rate of the central limit theorem (or the law of large numbers.) The large deviation principle has advantages in analysis to Stein's method, because the large deviation principle is related to the spectral analysis. On the other hand, Stein's method has advantages in computation and in practice, because the bound of the reminder term is obtained by explicit calculations. By using Stein's method, one can estimate the distances between the standard normal distribution and other distributions, where the distances mean, for example, Kolmogorov distance, Wasserstein distance, and total variation distance.

First we give the detail of Stein's equation and Stein's bound. Let Z be a random variable with the standard normal distribution and h be a measurable function on $\mathbb{R}$ such that $E[|h(Z)|] < \infty$. Stein's equation associated with h and Z is

$$h(x) - E[h(Z)] = f'(x) - xf(x), \quad x \in \mathbb{R}. \tag{5.1}$$

The solution f to (5.1) is obtained explicitly as follows:

$$f(x) = e^{\frac{1}{2}x^2} \int_{-\infty}^{x} (h(y) - E[h(Z)])\, e^{-\frac{1}{2}y^2}\, dy, \quad x \in \mathbb{R}. \tag{5.2}$$

By using (5.1) and (5.2), the following proposition holds.

Proposition 5.1. (i) *Let X be a random variable. Then, X has the standard normal distribution if and only if,*

$$E[f'(X) - Xf(X)] = 0$$

for every continuous and piecewise differentiable function f satisfying $E[|f'(Z)|] < \infty$.

(ii) *If* $h(x) = \mathbb{I}_{(-\infty,z]}(x)$ *where* $z \in \mathbb{R}$, *then the solution* f *to (5.1) exists,* f *is piecewise continuously differentiable,* $\|f\|_\infty \le \sqrt{2\pi}/4$, *and* $\|f'\|_\infty \le 1$.

(iii) *If* h *is bounded by* $1/2$, *the solution* f *to (5.1) exists,* f *is differentiable almost everywhere,* $\|f\|_\infty \le \sqrt{\pi/2}$, *and* $\|f'\|_\infty \le 2$.

(iv) *If* h *is bounded and absolutely continuous, then the solution* f *to (5.1) exists,* f *is bounded and twice differentiable,* $\|f\|_\infty \le \sqrt{\pi/2}\|h(y) - E[h(Z)]\|_\infty$, $\|f'\| \le 2\|h(y) - E[h(Z)]\|_\infty$, *and* $\|f''\|_\infty \le 2\|h'\|_\infty$.

(v) *If* h *is absolutely continuous and the derivative is bounded, then the solution* f *to (5.1) exists,* f *is twice differentiable,* $\|f'\|_\infty \le \|h'\|_\infty$, *and* $\|f''\|_\infty \le 2\|h'\|_\infty$.

The proofs of the statements in Proposition 5.1 are found in the references in [6].

By using (5.1) and the bound in Proposition 5.1, we can consider the estimate for the distances between the standard normal distribution and other distributions.

Consider a distance between distributions of random variables F and G on $\mathbb{R}$ defined by

$$d_{\mathcal{H}}(\mathcal{L}(F), \mathcal{L}(G)) := \sup_{f \in \mathcal{H}} |\mathrm{E}[f(F)] - \mathrm{E}[f(G)]|, \tag{5.3}$$

where $\mathcal{H}$ is a set of functions on $\mathbb{R}$. There are many distances between distributions defined by (5.3). For example, by taking $\mathcal{H} = \mathcal{F}_{\text{Kol}} := \{\mathbb{I}_{(-\infty,z]} : z \in \mathbb{R}\}$, one obtains the Kolmogorov distance; by taking $\mathcal{H} = \mathcal{F}_{\text{W}} := \{f : \|f\|_L \le 1\}$, where $\|\cdot\|_L$ denotes the usual Lipschitz seminorm, one obtains the Wasserstein (or Kantorovich-Wasserstein) distance; by taking $\mathcal{H} = \mathcal{F}_{\text{FM}} := \{f : \|f\|_{BL} \le 1\}$, where $\|\cdot\|_{BL} = \|\cdot\|_L + \|\cdot\|_\infty$, one obtains the Fortet-Mourier (or bounded Wasserstein) distance; by letting $\mathcal{H} = \mathcal{F}_{\text{TV}}$ be the collection of all indicators $\mathbb{I}_B$ of Borel sets, one obtains the total variation distance. We denote the Kolmogorov distance, the Wasserstein distance, Fortet-Mourier distance and the total variation distance by $d_{\text{Kol}}(\cdot,\cdot)$, $d_{\text{W}}(\cdot,\cdot)$, $d_{\text{FM}}(\cdot,\cdot)$ and $d_{\text{TV}}(\cdot,\cdot)$, respectively.

The following theorem is the result on the estimate of the distances, which is obtained by Nourdin and Peccati [6].

Theorem 5.2. *(Theorem 3.1 of [6]) Let* Z *has the standard normal distribution, and* $F \in \mathbb{D}^{1,2}$ *such that* $E[F] = 0$. *Then,*

$$d_{\text{W}}(F, Z) \le E\left[(1 - (DF, -DL^{-1}F)_H)^2\right]^{1/2},$$
$$d_{\text{FM}}(F, Z) \le E\left[(1 - (DF, -DL^{-1}F)_H)^2\right]^{1/2}.$$

If, in addition, the law of F *is absolutely continuous with respect to Lebesgue measure,*

$$d_{\text{Kol}}(F, Z) \le E\left[(1 - (DF, -DL^{-1}F)_H)^2\right]^{1/2},$$
$$d_{\text{TV}}(F, Z) \le 2E\left[(1 - (DF, -DL^{-1}F)_H)^2\right]^{1/2}.$$

On the other hand, by using Proposition 4.2, we have the following Proposition.

Proposition 5.3. *(Proposition 3.2 of [6]) Let* $n = 2, 3, 4, \ldots$, *and* $F = I_n(f)$ *where* $f \in H^{\odot n}$. *Then,* $(DF, -DL^{-1}F)_H = n^{-1}\|DF\|_H^2$, *and*

$$E\left[(1 - (DF, -DL^{-1}F)_H)^2\right]$$
$$= E\left[(1 - n^{-1}\|DF\|_H^2)^2\right] \tag{5.4}$$
$$= (1 - n!\|f\|_{H^{\otimes n}}^2)^2 + n^2 \sum_{r=1}^{n-1} (2n-2r)!\,[(r-1)!]^2 \binom{n-1}{r-1}^4 \|f \widetilde{\otimes}_r f\|_{H^{\otimes 2(n-r)}} \tag{5.5}$$
$$\le (1 - n!\|f\|_{H^{\otimes n}}^2)^2 + n^2 \sum_{r=1}^{n-1} (2n-2r)!\,[(r-1)!]^2 \binom{n-1}{r-1}^4 \|f \otimes_r f\|_{H^{\otimes 2(n-r)}}. \tag{5.6}$$

By using Theorem 5.2 and Proposition 5.3 we can simplify some parts of the proof of the fourth moment theorem (Theorem 4.3).

The case of the centered Gamma distribution is also considered in [6]. They prepared Stein's equation associated with the centered Gamma distribution, and obtained the bound of convergence to the centered Gamma distribution by similar way to the case of the standard normal distribution.

6 Further works on the fourth moment theorem and related topics

In this section we introduce some further studies around the fourth moment theorem.

6.1 The case of the centered Gamma distribution

An analogue of the fourth moment theorem to the centered Gamma distribution is obtained by Nourdin and Peccati [5]. The statement is as follows.

Let $\nu > 0$ and $G(\nu/2)$ be a random variable having the Gamma distribution with parameter $\nu/2$, i.e. $G(\nu/2)$ is a random variable with density function $g(x) = \frac{x^{\nu/2-1}e^{-x}}{\Gamma(\nu/2)}\mathbb{I}_{(0,\infty)}$, where Γ is the Gamma function. Consider a random variable $F(\nu)$ defined by

$$F(\nu) := 2G(\nu/2) - \nu.$$

The following theorem is an analogue of the fourth moment theorem with respect to $F(\nu)$.

Theorem 6.1. *Let $n \in 2\mathbb{N}$ and*

$$c_n := \frac{4}{(n/2)! \begin{pmatrix} n \\ n/2 \end{pmatrix}^2}.$$

Consider a sequence of random variables $G_k = I_n(g_k)$ where $g_k \in H^{\odot n}$ and assume that

$$\lim_{k\to\infty} E[G_k^2] = \lim_{k\to\infty} n!\|g_k\|^2_{H^{\otimes n}} = 2\nu.$$

Then, the following conditions are equivalent.

(i) *G_k converges to $F(\nu)$ in distribution as $k \to \infty$.*

(ii) *$\lim_{k\to\infty}(E[G_k^4] - 12E[G_k^3]) = 12\nu^2 - 48\nu$.*

(iii) *$\|DG_k\|_H^2 - 2nG_k$ converges to $2n\nu$ in $L^2(P)$ as $k \to \infty$.*

(iv) *$\lim_{k\to\infty} \|g_k\tilde{\otimes}_{n/2}g_k - c_n g_k\|_{H^{\otimes n}} = 0$ and $\lim_{k\to\infty} \|g_k \otimes_r g_k\|_{H^{\otimes 2(n-r)}} = 0$ for $r = 1, 2, \ldots, n-1$ except $r = n/2$.*

In [6], they discuss the case of the centered Gamma distribution in a similar way to the case of the standard normal distribution, and Stein's equation with respect to the centered Gamma distribution is obtained. We omit the version of Stein's equation in this note, because more general version of Stein's equation is in Section 6.2.

6.2 Generalization of Stein's bound

As we have seen in Theorem 5.2, by applying Malliavin calculus to Stein's equation we obtain the estimate of the distances between distributions. The cases of the standard normal distribution and the centered Gamma distribution are considered in [6], and more general argument is also mentioned as a conjecture in [6]. After Nourdin and Peccati [6], in [4] a general argument is constructed in view of the invariant measure of one-dimensional stochastic differential equation.

Let S be the interval (l, u) $(-\infty \leq l < u \leq \infty)$ and μ be a probability measure on S with a density function p which is continuous, bounded, strictly positive on S, and admits finite variance. Consider a continuous function b on S such that there exists $k \in (l, u)$ such that $b(x) > 0$ for $x \in (l, k)$ and $b(x) < 0$ for $x \in (k, u)$, bp is bounded on S and

$$\int_l^u b(x)p(x)dx = 0.$$

Define

$$a(x) := \frac{2\int_l^x b(y)p(y)dy}{p(x)}, \quad x \in S.$$

Then, the stochastic differential equation:

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t, \quad t \geq 0$$

has a unique Markovian weak solution, ergodic with invariant density p (see [1]).

For $f \in C_0(S)$ (the set of continuous functions on S vanishing at the boundary of S), let $m_f := \int_l^u f(x)p(x)dx$ and define $\tilde{g}_f$ by, for every $x \in S$,

$$\tilde{g}_f(x) := \frac{2}{a(x)p(x)} \int_l^x (f(y) - m_f)p(y)dy$$

Then, we have

$$\tilde{g}_f(x) = \int_l^x \frac{2(f(y) - m_f)}{a(y)} \exp\left(-\int_y^x \frac{2b(z)}{a(z)}dz\right)dy, \quad x \in S.$$

Then, $g_f(x) := \int_0^x \tilde{g}_f(y)dy$ satisfies that $f - m_f = Ag_f$ and

$$f(x) - \mathbf{E}[f(X)] = \frac{1}{2}a(x)\tilde{g}_f'(x) + b(x)\tilde{g}_f(x) \tag{6.1}$$

where X is a random variable with its law μ. The equation (6.1) is a generalized version of Stein's equation.

To obtain the estimate of the distances between the distribution associated with $p(x)dx$ and other distributions, we need the bounds for the functions $\tilde{g}_f$ and $\tilde{g}_f'$. Since we have the explicit form of $\tilde{g}_f$, the following propositions are obtained.

Proposition 6.2. *Assume that a is uniformly positive and there exist $l', u' \in (l, u)$ such that b is non-increasing on (l, l') and (u', u). Then we have*

$$||\tilde{g}_f||_\infty \leq C_1||f||_\infty \text{ and } ||a\tilde{g}_f'||_\infty \leq C_2||f||_\infty, \quad f \in C_0^\infty(S),$$

where C_1 and C_2 are strictly positive constants.

Proposition 6.3. *Assume that if $u < \infty$, there exists $u' \in (l, u)$ such that b is non-decreasing and Lipschitz continuous on $[u', u)$ and $\liminf_{x\to u} a(x)/(u - x) > 0$; if $u = \infty$, there exists $u' \in (l, u)$ such that b is non-decreasing on $[u', u)$ and $\liminf_{x\to u} a(x) > 0$. Similarly, assume that if $l > -\infty$, there exists $l' \in (l, u)$ such that b is non-increasing and Lipschitz continuous on $(l, l']$ and $\liminf_{x\to l} a(x)/(x - l) > 0$; if $l = -\infty$, there exists $l' \in (l, u)$ such that b is non-decreasing on $(l, l']$ and $\liminf_{x\to l} a(x) > 0$. Then we have*

$$||\tilde{g}_f'||_\infty \leq C_4(||f||_\infty + ||f'||_\infty), \quad f \in C_0^\infty(S),$$

where C_4 is a constant.

The estimates in Proposition 6.2 are sufficiently good when a is uniformly bounded and strictly positive. However, when a degenerates at the boundary of S, we need Proposition 6.3. We remark that in Proposition 6.3 the derivative of $\tilde{g}_f$ is dominated by the sum of $||f||_\infty$ and $||f'||_\infty$. In view of this fact it seems true that the case that a is uniformly positive and the case that a is degenerate are very different. In fact, the result obtained in the case of the standard normal distribution is different from that obtained in the case of the centered Gamma distribution (see Sections 5 and 6.1).

By using the generalized Stein's equation (6.1) and Propositions 6.2 and 6.3, the estimate of the distances between the distribution associated with $\mu = p(x)dx$ and other distributions is obtained in the same way as 5.

Theorem 6.4. (i) *Let d be the Fortet-Mourier distance. Assume the conditions in Proposition 6.3. Then,*

$$d(\mathcal{L}(Y), \mu) \leq CE\left[\left|\frac{1}{2}a(Y) + \langle D(-L)^{-1}\left\{b(Y) - \mathbf{E}[b(Y)]\right\}, DY\rangle_H\right|\right] + C|\mathbf{E}\left[b(Y)\right]|, \quad Y \in \mathbb{D}^{1,2}$$

where C is a positive constant and $\mathcal{L}(Y)$ is the law of Y.

(ii) *Let d be the Fortet-Mourier distance, the Kolmogorov distance or the total variation distance. Assume the conditions in Proposition 6.2 and a is uniformly positive. Then,*

$$d(\mathcal{L}(Y), \mu) \leq CE\left[\left|\frac{1}{2}a(Y) + \langle D(-L)^{-1}\left\{b(Y) - \mathbf{E}[b(Y)]\right\}, DY\rangle_H\right|\right] + C|\mathbf{E}\left[b(Y)\right]|, \quad Y \in \mathbb{D}^{1,2}$$

where C is a positive constant.

The bounds in Theorem 6.4 are optimal in the following sense.

Theorem 6.5. *A random variable $Y \in \mathbb{D}^{1,2}$ with its values on S has probability distribution μ and satisfies that $b(Y) \in L^2(\Omega)$ if and only if $E[b(Y)] = 0$ and*

$$E\left[\frac{1}{2}a(Y) + \langle D(-L)^{-1}b(Y), DY\rangle_H \middle| Y\right] = 0.$$

In Theorem 6.4, there is the term:

$$E\left[\left|\frac{1}{2}a(Y) + \langle D(-L)^{-1}\left\{b(Y) - \mathbf{E}[b(Y)]\right\}, DY\rangle_H\right|\right]. \tag{6.2}$$

Generally it is difficult to calculate this term. However, if a and b are given explicitly, and if Y is expressed as an explicit function of a Gaussian random variables, then (6.2) can be calculated by using the Ornstein-Uhlembeck semigroup and its resolvent (see [4] and [8].)

6.3 Other works

There are many other works on this topic.

The analogue in free probability theory has been concerned in [3]. In free probability theory, we also have the analogue of Wiener chaos (so-called Wigner chaos or free chaos). In [3], the analogues of the fourth moment theorem and Stein's bound with respect to the semicircular law are obtained.

In [2], the original Stein's method is applied to the theory of spin glasses. In the paper, the Thouless-Anderson-Palmer equations of the Sherrington-Kirkpatrick model is obtained. Moreover, the upper estimate of the convergence to the Thouless-Anderson-Palmer equations is also obtained by using the original Stein's method. Like this, the applications of Stein's method to statistical mechanics are also considered and some results have been obtained recently. We remark that the argument in the paper is away from the fourth moment theorem and the combination of Stein's method and the Malliavin calculus.

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Integrals along rough paths via fractional calculus

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In [2], Hu and Nualart introduced an alternative approach to the rough path analysis, and it was based on the fractional calculus. This approach has produced the integral along β-Hölder continuous functions of order $\beta \in (1/3, 1/2)$ by combining the ideas of the rough path analysis and the techniques of the integration by parts formula in terms of the fractional derivatives introduced by Zähle ([3]). This integral has given a new tool to study the differential equations driven by β-Hölder continuous functions of order $\beta \in (1/3, 1/2)$; for example a study on the stochastic differential equations driven by the fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2)$ is found in [1].

One of the interesting points of this approach is that the integral is given by way to the usual Lebesgue integrals based on the fractional derivatives, and the definition does not require any approximation arguments, unlike the integrals in the context of the rough path analysis, namely the rough integrals. It is expected that the further development of this approach would provide a sophisticated access to the fundamental theory of the rough path analysis. Therefore, we arrive at the natural question to consider next, that is whether this approach is valid for any $\beta \in (0, 1]$; in particular, it drives us to the following question.

Question. Can the rough integrals be expressed as usual Lesbesgue integrals based on the fractional derivatives for any $\beta \in (0, 1]$?

In this talk, we will give an affirmative answer to this question, and produce the integral along β-Hölder rough paths based on the fractional derivatives for any $\beta \in (0, 1]$. It is a generalization of the preceding study of [2] in the following sense. This integral is a natural generalization of the Riemann–Stieltjes integral along smooth curves, and a continuous functional with respect to the β-Hölder topology under suitable conditions on the integrand. Consequently, we will obtain the following answer to the above question.

Answer. The first level path of the rough integrals along the geometric β-Hölder rough paths can be expressed as usual Lesbesgue integrals based on the fractional derivatives for any $\beta \in (0, 1]$.

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Functional limit theorem for processes pieced together from excursions

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A sample path of a process can be constructed by being pieced together from excursions away from a prescribed point a. Itô's excursion theory assures that, for a strong Markov process where the point a is regular for itself, the point process of excursions is Poisson and hence is characterized by its characteristic measure called the *excursion measure.*

In [1], the author obtained some homogeneity results for jumping-in diffusion processes. The proof was based on the construction of a sample path from excursions and the functional convergence of the suitable scaling for the pieced process was proved via that for excursions.

In this talk, we provide a more general framework for proving convergence of the pieced process from that of excursions. We assume that the excursion measures considered are realized as pullback of a common measure by measurable maps and those maps converge in the function space. We then prove convergence of the pieced process in law on the Skorokhod space.

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Variance mixture and subordinator in free probability

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Abstract

First, we consider infinitely divisibility of free counterpart of variance mixture of normal distribution. From this problem, we find a class of the freely infinitely divisible distributions that appear as the laws of free subordinators play an important role to investigate infinite divisibility. We call this class free regular infinitely divisible measures. We prove that the class of free regular measures is closed under the free multiplicative convolution, t-th boolean power for $0 < t < 1$, t-th free multiplicative power for $t > 1$ and weak convergence. In addition, we show that a symmetric distribution is freely infinitely divisible if and only if its square can be represented as the free multiplicative convolution of a free Poisson and a free regular measure. This gives two new explicit examples of distributions which are infinitely divisible with respect to both classical and free convolutions: chi-square and $F(1;1)$-distribution.

APPROXIMATING ROUGH STOCHASTIC PDES

JAN MAAS

We study a class of vector-valued equations of Burgers type driven by a multiplicative space-time white noise. These equations are of the form

$$\partial_t u = \nu\, \partial_x^2 u + F(u) + G(u)\partial_x u + \theta(u)\, \xi\ , \tag{1}$$

where the function $u = u(t, x; \omega) \in \mathbb{R}^n$ is vector-valued. We assume that the functions $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and $G, \theta\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are smooth and the products in the terms $G(u)\partial_x u$ as well as in $\theta(u)\xi$ are to be interpreted as matrix vector multiplication. The noise term ξ denotes an $\mathbb{R}^n$-valued space-time white noise and the multiplication should be interpreted in the sense of Itô integration against an L^2-cylindrical Wiener process.

In the case where G is the gradient of a function $\mathcal{G}$ the equation (1) is classically well-posed. The definition of weak solutions and their construction uses the conservation law structure of (1): The nonlinearity is rewritten as

$$G(u)\partial_x u = \partial_x \mathcal{G}(u),$$

and the derivative can be treated by integration by parts. However, several seemingly natural approximation schemes fail to produce solutions of (1), but converge to different limit equations in which extra terms may appear.

In the case where G is not a total derivative it is not even clear how to make sense of (1). The solution does not have the regularity required to make sense of the nonlinearity. We use rough path theory to resolve this issue. Weak solutions can be defined by testing against a smooth test function φ and defining the term

$$\int_{-\pi}^{\pi} \varphi(x) G(u(t,x))\partial_x u(t,x) dx$$

as a rough integral.

We study approximations to (1) of the form

$$du_\varepsilon = \Big(\nu \Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + G(u_\varepsilon) D_\varepsilon u_\varepsilon\Big) dt + \theta(u_\varepsilon)\, H_\varepsilon dW,$$

for a large class of regularisations $\Delta_\varepsilon, D_\varepsilon$, and H_ε. We show that the u_ε converge to a process $\bar{u}$ that solves an equation similar to (1) with an extra term

$$-\Lambda\, \theta(u) \nabla G(u)\, \theta^T(u).$$

This term is the local spatial cross variation of u and $G(u)$ and can be interpreted as a spatial Itô-Stratonovich correction. The constant Λ depends on the specific choice of the approximations and can be calculated explicitly. We obtain a rate of convergence of $\varepsilon^{1/6}$.

This is joint work with Martin Hairer and Hendrik Weber.

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Asymptotics of quantum walks on the line

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The notion of quantum walks, often called discrete time quantum random walks, was introduced by Aharonov-Davidovich-Zagury ([ADZ]) in 1993 as a quantum analogue of the classical random walks, and re-discovered in the area of computer science. In particular, Ambainis-Kempe-Rivosh ([AKR]) utilized two-dimensional quantum walks to improve Grover's quantum search algorithm. In the talk, various local asymptotic formulas of transition probabilities of quantum walks on the one-dimensional integer lattice, obtained in [ST], will be given. In the present article, we just mention one of the formulas, which is a limit formula of a large deviation type. To be precise, let us give a definition of quantum walks on the one-dimensional integer lattice. The quantum walks we consider in the talk is defined by a (special) unitary matrix,

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1,$$

and its decomposition,

$$A = P + Q, \quad P = \begin{pmatrix} a & 0 \\ -\bar{b} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & b \\ 0 & \bar{a} \end{pmatrix}.$$

Let $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ be the Hilbert space of square summable functions on $\mathbb{Z}$ with values in $\mathbb{C}^2$ whose inner product is given by

$$\langle f, g \rangle = \sum_{x \in \mathbb{Z}} \langle f(x), g(x) \rangle_{\mathbb{C}^2}, \quad f, g \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ denotes the standard inner product on $\mathbb{C}^2$. For any $u \in \mathbb{C}^2$ and $x \in \mathbb{Z}$, define $\delta_x \otimes u \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$(\delta_x \otimes u)(y) = \begin{cases} u & (y = x), \\ 0 & (y \neq x). \end{cases}$$

Then, the vectors, $\delta_x \otimes \mathrm{e}_i$ ($i = 1, 2,\ x \in \mathbb{Z}$), where $\{\mathrm{e}_1, \mathrm{e}_2\}$ is the standard orthonormal basis in $\mathbb{C}^2$, form an orthonormal basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. The unitary evolution, U, of the quantum walks on $\mathbb{Z}$ is a unitary operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined as

$$U = P\tau^{-1} + Q\tau,$$

where τ is the shift operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined by $\tau(\delta_x \otimes u) = \delta_{x+1} \otimes u$. The operator U is indeed a unitary operator, and hence the function

$$p_n(\varphi; x) = \|U^n(\delta_0 \otimes \varphi)(x)\|^2_{\mathbb{C}^2}, \quad x \in \mathbb{Z}$$

defines a probability distribution on $\mathbb{Z}$ supported on $[-n, n]$ for any unit vector φ in $\mathbb{C}^2$ and positive integer n, which we call the transition probability of the quantum walk U. The behavior of $p_n(\varphi; x)$ as $n \to \infty$ is one of main topics in the study of quantum walks. Indeed, as the following Figure 1[†] shows, it is drastically different from the behavior of transition probabilities of classical random walks. In Figure 1, the 'wall' is located at $x/n \sim \pm|a|$, where a is a component of the given unitary matrix A. The behavior of $p_n(\varphi; x)$ heavily depends on the 'normalized' position x/n according as

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[†]Figure 1 is due to Dr. Takuya Machida in Meiji University.

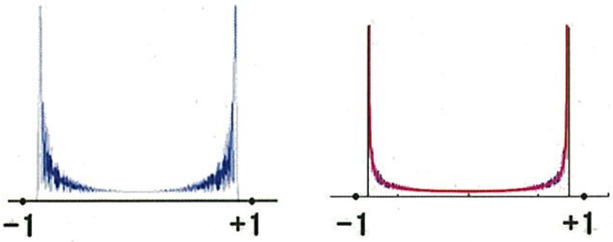

Figure 1: Probability disctribution and its weak-limit distribution

(1) x/n is inside the interval, $(-|a|, |a|)$,

(2) x/n stays around the 'wall', say, $x/n \sim \pm|a|$, or

(3) x/n is outside the interval, say, $|x/n| > |a|$.

Our analysis in [ST] gives precise asymptotic formulas in each region (1) – (3). For instance, a corollary to our results is stated as follows.

Corollary *Let $\xi \in \mathbb{R}$ satisfy $|a| < |\xi| < 1$. Suppose that a sequence of integers, $\{x_n\}$, satisfies*

$$x_n = n\xi + O(1) \quad (n \to \infty).$$

If $p_n(\varphi; x_n) \neq 0$ for every sufficiently large n, we have the following limit formula of a large deviation type,

$$\lim_{n\to\infty} \frac{1}{n} p_n(\varphi; x_n) = -H_Q(\xi),$$

where the function $H_Q(\xi)$ is given by

$$H_Q(\xi) = 2|\xi| \log\left(|b||\xi| + \sqrt{\xi^2 - |a|^2}\right) - 2\log\left(|b| + \sqrt{\xi^2 - |a|^2}\right) + (1 - |\xi|)\log(1 - \xi^2) - 2|\xi|\log|a|.$$

In the talk, after an explanation of backgrounds, properties and known results, such as a weak limit formula due to Konno ([K]), on the quantum walks on $\mathbb{Z}$ comparing with classical random walks, our main results on the asymptotic formulas of $p_n(\varphi; x)$ are introduced. According to our results, the asymptotic behavior of $p_n(\varphi; x)$ has indeed a quantum mechanical nature. The resemblance of the asymptotic behavior of $p_n(\varphi; x)$ and that of the Hermite functions will be pointed out by introducing the Plancerel-Rotach formula on asymptotic behavior of the Hermite functions.

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Exponential convergence of Markovian semigroups*

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1 Hypercontractivity and the exponential convergence

Let $(M, \mathscr{B}, m)$ be a measure space with $m(M) = 1$. Suppose we are given a Markovian semigroup $\{T_t\}$ in $L^2(m)$. We denote its dual semigroup $\{T_t^*\}$ and assume that $\{T_t^*\}$ is Markovian and $T_t 1 = 1$ and $T_t^* 1 = 1$. $\{T_t\}$ and $\{T_t^*\}$ define strongly continuous semigroups in $L^p(m)$ $(1 \leq p < \infty)$ naturally.

We are interested in the following ergodicity:

$$T_t f \to \langle f \rangle \quad \text{as } t \to \infty$$

To be precise, define the index $\gamma_{p \to q}$ by

$$\gamma_{p \to q} = -\overline{\lim} \frac{1}{t} \log \|T_t - m\|_{p \to q}. \tag{1}$$

Here m denotes an operator $f \mapsto m(f) = \int_X f \, dm$ and $\| \ \|_{p \to q}$ denotes the operator norm from L^p to L^q.

We recall that $\{T_t\}$ is called hyperbounded if there exist $K > 0$, $r \in (2, \infty)$ and $C \geq 1$ such that

$$\|T_K f\|_r \leq C \|f\|_2, \quad \forall f \in L^2(m).$$

Then we have

Theorem 1. *The followings are equivalent to each other:*

(1) $\{T_t\}$ *is hyperbounded.*

(2) $\gamma_{p \to q} \geq 0$ *for some* $1 < p < q < \infty$.

(3) $\gamma_{p \to q} = \gamma_{2 \to 2}$ *for all* p, $q \in (1, \infty)$.

Also $\{T_t\}$ is called hypercontractive if there exist $K > 0$ and $r \in (2, \infty)$ such that

$$\|T_K f\|_r \leq \|f\|_2, \quad \forall f \in L^2(m).$$

Then we have

Theorem 2. *The followings are equivalent to each other:*

(1) $\{T_t\}$ *is hypercontractive.*

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(2) $\gamma_{p\to q} > 0$ *for some* $1 < p < q < \infty$.

(3) $\gamma_{p\to q} = \gamma_{2\to 2} > 0$ *for all* p, $q \in (1, \infty)$.

Further if we assume that the generator $\mathfrak{A}$ of T_t is normal, we have the following p-independence of the spectrum.

Theorem 3. *Assume* $\mathfrak{A}$ *is normal. Then* $\sigma(\mathfrak{A}_p)$, *the spectrum of* $\mathfrak{A}_p$, *is independent of* p *(*$1 < p < \infty$*).*

2 Example of L^p-spectrum that depends on p

We give an example that the spectrum depends on p. Let $M = [0, \infty)$ and $m(dx) = \nu(dx) = e^{-x}dx$. The Dirichlet form in $L^2(\nu)$ is given by

$$\mathscr{E}(f, g) = \int_{[0,\infty)} f'(x)g'(x)\nu(dx).$$

The generator is

$$\mathfrak{A} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with boundary condition $f'(0) = 0$.

Theorem 4. *For* $p = 2$, *we have*

$$\sigma(-\mathfrak{A}) = \{0\} \cup [\frac{1}{4}, \infty).$$

Theorem 5. *For* $1 \le p < 2$, *we have*

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}) = \{0\} \cup \{x + iy;\ x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(ii) $\sigma_{\mathrm{c}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\rho(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

Theorem 6. *For* $p > 2$, *we have*

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}) = \{0\}$

(ii) $\sigma_{\mathrm{r}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\sigma_{\mathrm{c}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(iv) $\rho(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

Historical Superprocess Related to Random Measure

ランダム測度に付随するヒストリカル超過程

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1. Superprocess with Branching Rate Functional

We introduce the superprocess with branching rate functional, which forms a general class of measure-valued branching Markov processes with diffusion as a underlying spatial motion. We write as $\langle \mu, f\rangle = \int f d\mu$. For simplicity, $M_F = M_F(\mathbb{R}^d)$ is the space of finite measures on $\mathbb{R}^d$. Define a second order elliptic differential operator $L = \frac{1}{2}\nabla \cdot a\nabla + b\cdot\nabla$, and $a = (a_{ij})$ is positive definite and we assume that $a_{ij}, b_i \in C^{1,\varepsilon} = C^{1,\varepsilon}(\mathbb{R}^d)$. Here $C^{1,\varepsilon}$ indicates the totality of all Hölder continuous functions with index ε $(0 < \varepsilon \leqslant 1)$, allowing their first order derivatives to be locally Hölder continuous. $\{\xi, \Pi_{s,a}\}$ indicates a corresponding L-diffusion. Moreover CAF stands for continuous additive functional. Let $\mathbb{K}^q$ (with $q > 0$) denote the Dynkin class of locally admissible CAF's with index q. When we write C_b as the set of bounded continuous functions on $\mathbb{R}^d$, then C_b^+ is the set of positive members in C_b. The process $\{X, \mathbb{P}_{s,\mu}\}$ is said to be a *superprocess with branching rate functional* K or simply (L, K, μ)-*superprocess* if X is a continous M_F-valued time-inhomogeneous Markov process with Laplace functional $\mathbb{P}_{s,\mu} e^{-\langle X_t,\varphi\rangle} = e^{-\langle \mu, v(s,t)\rangle}$ for $0 \leqslant s \leqslant t$, $\mu \in M_F$ and $\varphi \in C_b^+$. Here v is uniquely determined by the log-Laplace equation

$$\Pi_{s,a}\varphi(\xi_t) = v(s,a) + \Pi_{s,a}\int_s^t v^2(r,\xi_r)K(dr), \qquad 0 \leqslant s \leqslant t, \quad a \in \mathbb{R}^d. \tag{1}$$

2. Associated Historical Superprocess

The historical superprocess was initially studied by Dynkin (1991) (cf. Dawson-Perkins, 1991). $\mathbb{C}$ denotes the space of continuous paths on $\mathbb{R}^d$ with topology of uniform convergence. To each $w \in \mathbb{C}$ and $t > 0$, we write $w^t \in \mathbb{C}$ as the stopped path of w. We denote by $\mathbb{C}^t$ the totality of all these paths stopped at t. To every $w \in \mathbb{C}$ we associate the corresponding stopped path trajectory $\tilde{w}$ defined by $\tilde{w}_t = w^t$ $(t \geq 0)$. The image of L-diffusion w under the map : $w \mapsto \tilde{w}$ is called the *L-diffusion path process.* Moreover, we define $\mathbb{C}_R^\times = \{(s,w) : s \in \mathbb{R}_+, w \in \mathbb{C}^s\}$ and we denote by $M(\mathbb{C}_R^\times)$ the set of measures η on $\mathbb{C}_R^\times$ which are finite, if restricted to a finite time interval. Let K be a positive CAF of ξ. $\{\tilde{X}, \tilde{\mathbb{P}}_{s,\mu}\}$ is said to be a Dynkin's *historical superprocess* if $\tilde{X}$ is a time-inhomogeneous Markov process with state $\tilde{X}_t \in M_F(\mathbb{C}^t)$, $t \geq s$, with Laplace functional $\tilde{\mathbb{P}}_{s,\mu} e^{-\langle \tilde{X}_t,\varphi\rangle} = e^{-\langle \mu, v(s,t)\rangle}$

for $0 \leqslant s \leqslant t$, $\mu \in M_F(\mathbb{C}^s)$ and $\varphi \in C_b^+(\mathbb{C})$, where v is uniquely determined by the log-Laplace equation

$$\tilde{\Pi}_{s,w_s}\varphi(\tilde{\xi}_t) = v(s,w_s) + \tilde{\Pi}_{s,w_s}\int_s^t v^2(r,\tilde{\xi}_r)\tilde{K}(dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \qquad (2)$$

We call this $\tilde{X}$ an associted historical superprocess in Dynkin's sense.

3. Superprocess Related to Random Measure

Suppose that $p > d$, and let $\phi_p(x)$ be the reference function. C denotes the space of continuous functions on $\mathbb{R}^d$, and define $C_p = \{f \in C : |f| \leqslant C_f \cdot \phi_p, \exists C_f > 0\}$. We denote by M_p the set of non-negative measures μ on $\mathbb{R}^d$, satisfying $\langle \mu, \phi_p \rangle < \infty$. It is called the space of p-tempered measures. We define the continuous additive functional K_η of ξ by $K_\eta = \langle \eta, \delta_x(\xi_r)\rangle dr$ for $\eta \in M_p$. Then $X^\eta = \{X_t^\eta; t \geq 0\}$ is said to be a measure-valued diffusion with branching rate functional K_η if for $\mu \in M_F$, X satisfies the Laplace functional $\mathbb{P}^\eta_{s,\mu} e^{-\langle X_t^\eta, \varphi\rangle} = e^{-\langle \mu, v(s)\rangle}$ for $\varphi \in C_b^+$, where the function v is uniquely determined by

$$\Pi_{s,a}\varphi(\xi_t) = v(s,a) + \Pi_{s,a}\int_s^t v^2(r,\xi_r)K_\eta(dr), \quad (0 < s \leqslant t, a \in \mathbb{R}^d). \qquad (3)$$

Assume that $d = 1$ and $0 < \nu < 1$. Let $\lambda \equiv \lambda(dx)$ be the Lebesgue measure on $\mathbb{R}$, and let $(\gamma, \mathbb{P})$ be the stable random measure on $\mathbb{R}$ with Laplace functional

$$\mathbb{P}e^{-\langle \gamma, \varphi\rangle} = \exp\left\{-\int \varphi^\nu(x)\lambda(dx)\right\}, \quad \varphi \in C_b^+. \qquad (4)$$

Let $p > \nu^{-1}$ in what follows. We consider a positive CAF $K_{\gamma(\omega)}$ of ξ for $\mathbb{P}$-a.a. ω. So that, thanks to Dynkin's general formalism, there exists an (L, K_γ, μ)-superprocess X^γ when we adopt a p-tempered measure γ for K_η instead of η, as far as K_γ may lie in $\mathbb{K}^q$ $(\exists q > 0)$.

4. Historical Superprocess Related to Random Measure

As for the historical superprocess associated with the superprocess X^γ related to random measure, we can prove the following.

THEOREM. (Main Result) *Let K_γ be a positive CAF of ξ lying in the Dynkin class $\mathbb{K}^q$. Then there exists a historical superprocess $\tilde{\mathbb{X}}^\gamma = \{\tilde{X}^\gamma, \tilde{\mathbb{P}}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^s)\}$ in the Dynkin sense. In fact, $\tilde{\mathbb{X}}^\gamma$ is a time-inhomogeneous Markov process with state $\tilde{X}_t^\gamma \in M_F(\mathbb{C}^t)$, $t \geq s$, with Laplace functional $\tilde{\mathbb{P}}^\gamma_{s,\mu}\exp\{-\langle \tilde{X}_t^\gamma, \varphi\rangle\} = e^{-\langle \mu, v(s,t)\rangle}$ for $0 \leqslant s \leqslant t$, $\mu \in M_F(\mathbb{C}^s)$ and $\varphi \in C_b^+(\mathbb{C})$, where v is uniquely determined by the log-Laplace equation*

$$\tilde{\Pi}_{s,w_s}\varphi(\tilde{\xi}_t) = v(s,w_s) + \tilde{\Pi}_{s,w_s}\int_s^t v^2(r,\tilde{\xi}_r)\tilde{K}_\gamma(\omega; dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \quad (5)$$

A Clark-Ocone type formula under change of measure for Lévy processes

Ryoichi Suzuki*

The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives that turns to be central in the application to mathematical finance. In this talk, we introduce a Clark-Ocone type formula under change of measure for Lévy processes with L^2-Lévy measure ([5, 6]). As an application of the theorem, we are also preparing a paper concerning the local risk minimization problem ([1]).

Throughout this talk, we consider Malliavin calculus for Lévy processes, based on, [4] and [2]. Let $X = \{X_t; t \in [0,T]\}$ be a centered square integrable Lévy process with representation

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmented filtration generated by X and σ is a constant number. Furthermore, we assume that $\{W_t; t \in [0,T]\}$ is a standard Brownian motion and that N is a Poisson random measure independent of W defined by

$$N(A,t) = \sum_{s\leq t} 1_A(\Delta X_s), A \in \mathcal{B}(\mathbb{R}_0), \Delta X_s := X_s - X_{s-},$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. In addition, we will denote by $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ the compensated Poisson random measure, where $dt\nu(dz) = \lambda(dt)\nu(dz)$ is the compensator of N, $\nu(\cdot)$ the Lévy measure of X and λ the Lebesgue measure on $\mathbb{R}$. Since X is square integrable, the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Now we consider a finite measure q defined on $[0,T] \times \mathbb{R}$ by

$$q(E) = \sigma^2 \int_{E(0)} dt + \int_{E'} z^2\nu(dz)dt, \quad E \in \mathcal{B}([0,T] \times \mathbb{R}),$$

where $E(0) = \{t \in [0,T]; (t,0) \in E\}$ and $E' = E - E(0)$, and a random measure Q on $[0,T] \times \mathbb{R}$ by

$$Q(E) = \sigma \int_{E(0)} dW(t) + \int_{E'} z\tilde{N}(dt,dz), \quad E \in \mathcal{B}([0,T] \times \mathbb{R}).$$

Let $L^2_{T,q,n}(\mathbb{R})$ denote a set of product measurable, deterministic functions $f : ([0,T] \times \mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|f\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |f((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n \in \mathbb{N}$ and $f_n \in L^2_{T,q,n}(\mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q can be defined as

$$I_n(f_n) := \int_{([0,T]\times\mathbb{R})^n} f_n((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

In this setting, we introduce the following chaos expansion (see Theorem 2 in [3], Section 2 of [4]).

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Proposition 1 *Any $\mathcal{F}$-measurable square integrable random variable F has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \mathbb{P}-\text{a.s.}$$

with functions $f_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i, z_i), 1 \le i \le n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2_{T,q,n}}.$$

We next define the follows:

Definition 1 1. *Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying*

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2_{T,q,n}} < \infty.$$

2. *Let $F \in \mathbb{D}^{1,2}(\mathbb{R})$. Then the Malliavin derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}(\mathbb{R})$ is a stochastic process defined by*

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q-\text{a.e. } (t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s.$$

3. *Let $\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 q(ds,dx)\right] < \infty,$$

$G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), q-$a.e. $(s,x) \in [0,T] \times \mathbb{R}$ *and*

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,x)|^2 q(ds,dx) q(dt,dz)\right] < \infty.$$

4. *Let $\mathbb{L}^{1,2}_0(\mathbb{R})$ denote the space of measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]} |G(s)|^2 ds\right] < \infty,$$

$G(s) \in \mathbb{D}^{1,2}(\mathbb{R}),\ s \in [0,T],$ a.e. *and*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]} |D_{t,z}G(s)|^2 ds\, q(dt,dz)\right] < \infty.$$

5. *Let $\tilde{\mathbb{L}}^{1,2}_1(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|^2 \nu(dx) ds\right] < \infty,$$

$$\mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G(s,x)| \nu(dx) ds\right)^2\right] < \infty,$$

$G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s,x) \in [0,T] \times \mathbb{R}_0$, a.e. ,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)|\nu(dx)ds\right)^2 q(dt,dz)\right] < \infty$$

and

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)|^2\nu(dx)dsq(dt,dz)\right] < \infty.$$

Now, we assume the following.

Assumption 1 *Let $\theta(s,x) < 1, s \in [0,T], x \in \mathbb{R}_0$ and $u(s), s \in [0,T]$, be predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0} \{|\log(1-\theta(s,x))| + \theta^2(s,x)\}\nu(dx)ds < \infty, \text{ a.s.,}$$
$$\int_0^T u^2(s)ds < \infty, \text{ a.s.}$$

Moreover we denote

$$\begin{aligned} Z(t) \quad := \quad & \exp\left(-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u(s)^2 ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\tilde{N}(ds,dx)\right. \\ & \left. + \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds\right), t \in [0,T]. \end{aligned}$$

Define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q}(\omega) = Z(\omega,T)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(s)ds + \int_0^T \int_{\mathbb{R}_0} \{(1-\theta(s,x))\log(1-\theta(s,x)) + \theta(s,x)\}\nu(dx)ds\right)\right] < \infty.$$

Furthermore we denote

$$\tilde{N}_{\mathbb{Q}}(dt,dx) := \theta(t,x)\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_{\mathbb{Q}}(t) := u(t)dt + dW(t).$$

Second, we assume the following.

Assumption 2 *We denote*

$$\begin{aligned} \tilde{H}(t,z) \quad := \quad & \exp\left(-\int_0^T zD_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T (zD_{t,z}u(s))^2 ds\right. \\ & + \int_0^T \int_{\mathbb{R}_0} \left[zD_{t,z}\theta(s,x) + \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x))\right]\nu(dx)ds \\ & \left. + \int_0^T \int_{\mathbb{R}_0} \tilde{N}_{\mathbb{Q}}(ds,dx)\right), \end{aligned}$$

and

$$K(t) := \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds,dx).$$

Furthermore, assume that the following:

1. $F, Z(T) \in \mathbb{D}^{1,2}(\mathbb{R})$, *with* $(F + Z(T))^2 \in L^2(\mathbb{P})$,

$$(F + Z(T))(D_{t,z}F + D_{t,z}Z(T)), z(D_{t,z}F + D_{t,z}Z(T))^2 \in L^2(q \times \mathbb{P}),$$

2. $Z(T)D_{t,0} \log Z(T) \in L^2(\lambda \times \mathbb{P}), Z(T)(e^{zD_{t,z} \log Z(T)} - 1) \in L^2(\nu(dz)dtd\mathbb{P})$,
3. $u(s)D_{t,0}u(s) \in L^2(\lambda \times \mathbb{P}), 2u(s)D_{t,z}u(s) + z(D_{t,z}u(s))^2 \in L^2(z^2\nu(dz)dtd\mathbb{P})$, *s-a.e.*
4. $\log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right) \in L^2(\nu(dz)dtd\mathbb{P}), \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} \in L^2(\lambda \times \mathbb{P}), (s,x)$ *-a.e.*
5. $u, x^{-1}\log(1 - \theta(s,x)) \in \mathbb{L}^{1,2}(\mathbb{R})$,
6. $u(s)^2 \in \mathbb{L}_0^{1,2}$ *and* $\theta(s,x), \log(1 - \theta(s,x)) \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$,
7. *and* $F\tilde{H}(t,z), \tilde{H}(t,z)D_{t,z}F \in L^1(\mathbb{Q}), (t,z)$ *-a.e.*

We next introduce a Clark-Ocone type formula under change of measure for Lévy processes.

Theorem 1 *Under Assumption 1 and Assumption 2, we have*

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{Q}}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz). \end{aligned}$$

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Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians*

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Let $-L+V_\lambda$ be a spatially cut-off $P(\phi)_2$-Hamiltonian, where $\lambda = 1/\hbar$ is a large positive parameter. The operator $-L$ is the free Hamiltonian, that is the second quantization operator of $\sqrt{m^2-\Delta}$, where m is a positive number. The potential function V_λ is given by a Wick polynomial

$$V_\lambda(w) = \lambda \int_{\mathbb{R}} : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx, \tag{1}$$

where g is a smooth cut-off function and $P(x) = \sum_{k=1}^{2M} a_k x^k$ is a polynomial bounded from below. Formally, $-L_A + V_\lambda$ is unitarily equivalent to the infinite dimensional Schrödinger operator:

$$-\Delta_{L^2(\mathbb{R})} + \lambda U(w/\sqrt{\lambda}) - \frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2} \quad \text{on } L^2(L^2(\mathbb{R}), dw) \tag{2}$$

where dw is an infinite dimensional Lebesgue measure. The function U is a potential function such that

$$U(w) = \frac{1}{4}\int_{\mathbb{R}} w'(x)^2 dx + \int_{\mathbb{R}} \left(\frac{m^2}{4}w(x)^2 + : P(w(x)) : g(x)\right) dx$$

and $\Delta_{L^2(\mathbb{R})}$ denotes the "Laplacian"on $L^2(\mathbb{R}, dx)$. Hence, by the analogy of Schrödinger operators in finite dimensions, it is natural to expect that asymptotic behavior of lowlying eigenvalues of the spatially cut-off $P(\phi)_2$-Hamiltonian in the semiclassical limit $\lambda \to \infty$ is related with the global minimum points of U. In view of this, we consider the following assumptions.

Assumption 1. Let P be the polynomial in (1) and U be the function on H^1 which is given by

$$U(h) = \frac{1}{4}\int_{\mathbb{R}} h'(x)^2 dx + \int_{\mathbb{R}} \left(\frac{m^2}{4}h(x)^2 + P(h(x))g(x)\right) dx \qquad \text{for } h \in H^1. \tag{3}$$

(A1) The function U is non-negative and the zero point set

$$\mathcal{Z} := \{h \in H^1 \mid U(h) = 0\} = \{h_1, \ldots, h_n\} \tag{4}$$

is a finite set.
(A2) For all $1 \le i \le n$, the Hessian $\nabla^2 U(h_i)$ is non-degenerate. That is, there exists $\delta_i > 0$ for each i such that

$$\begin{aligned}\nabla^2 U(h_i)(h,h) &:= \frac{1}{2}\int_{\mathbb{R}} h'(x)^2 dx + \int_{\mathbb{R}} \left(\frac{m^2}{2}h(x)^2 + P''(h_i(x))g(x)h(x)^2\right) dx \\ &\ge \delta_i \|h\|^2_{L^2(\mathbb{R})} \qquad \text{for all } h \in H^1(\mathbb{R}).\end{aligned} \tag{5}$$

(A3) For all x, $P(x) = P(-x)$ and $\mathcal{Z} = \{h_0, -h_0\}$, where $h_0 \neq 0$.

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Let $E_1(\lambda)$ be the lowest eigenvalue of $-L + V_\lambda$. The first main result is as follows.

Theorem 2. Assume that (A1) and (A2) hold. Let $E_1(\lambda) = \inf \sigma(-L + V_\lambda)$. Then

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\le i\le n} E_i, \tag{6}$$

where

$$E_i = \inf \sigma(-L + Q_{v_i}) \tag{7}$$

and Q_{v_i} is given by

$$Q_{v_i}(w) = \int_{\mathbb{R}} : w(x)^2 : v_i(x)dx, \quad v_i(x) = \frac{1}{2}P''(h_i(x))g(x). \tag{8}$$

Remark 3. In the case of finite dimensional Schrödinger operators, there exist eigenvalues near the approximate eigenvalues E_i when λ is large. In Theorem 2, if $E_i < m + \min_{1\le i\le n} E_i$, then the same results hold by the result of Hoegh-Krohn and Simon. However, if it is not the case, it is not clear and they may be embedded eigenvalues in the essential spectrum. Under the assumptions in Theorem 5, $E_2(\lambda)$ is an eigenvalue for large λ.

Let

$$E_2(\lambda) = \inf \{\sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\}\}.$$

We can prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when U is a symmetric double well potential function. The exponential decay rate is given by the Agmon distance which is defined below.

Definition 4. Let $0 < T < \infty$ and $h, k \in H^1(\mathbb{R})$. Let $AC_{T,h,k}(H^1(\mathbb{R}))$ be the all absolutely continuous paths $c : [0,T] \to H^1(\mathbb{R})$ satisfying $c(0) = h, c(T) = k$. Let U be the potential function in (3). Assume U is non-negative. We define the Agmon distance between h, k by

$$d_U^{Ag}(h,k) = \inf \left\{\ell_U(c) \mid c \in AC_{T,h,k}(H^1(\mathbb{R}))\right\}, \tag{9}$$

where

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2}dt. \tag{10}$$

The following estimate is the second main result.

Theorem 5. Assume that U satisfies (A1),(A2),(A3). Then it holds that

$$\limsup_{\lambda\to\infty} \frac{\log (E_2(\lambda) - E_1(\lambda))}{\lambda} \le -d_U^{Ag}(h_0, -h_0). \tag{11}$$