Local Smoothness of the Densities of Solutions of SDEs with Singular Coefficients *

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1 Introduction

Consider the following one dimensional SDE of the form

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \ (0 \le t \le T), \tag{1}$$

where $x_0 \in \mathbb{R}$ and $(B)_{t\ge 0}$ is a one dimensional Brownian motion.

Note that if we assume that the coefficients of a hypoelliptic SDE are bounded functions with bounded derivatives of any order, then the solution of (1) has a smooth density (see, for example, Nualart[4]). In recent years, one of the directions in this area is to develop tools to deal with the case of non-smooth coefficients.

Some related results have already been obtained for this problem, for example, Fournier and Printems [1] proved in the case that σ is α-Hölder continuous with $\alpha > \frac{1}{2}$ and b is at most linear growth then the density of X_t exists for all $t > 0$. In that case, they showed the existence of the density on the set $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$. A careful analysis of their method shows that it is not amenable to obtain any further properties of the density (such as Hölder continuity).

For the multi-dimensional SDEs whose coefficients depends on time, Kusuoka [2] introduced some special space denoted by V_h which is larger than Sobolev space and showed the relation between the space V_h and absolute continuity. According to [2], one can show that the existence of the density of X_t on the set $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$ when the coefficients are bounded, σ is twice continuously differentiable on $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$ and b is Lipschitz continuous on $\mathbb{R}$.

2 Main Result

Definition 1. *Let $y_0 \in \mathbb{R}$ and $\varepsilon > 0$. The law of X has a density function p_{y0} on $B_\varepsilon(y_0) := \{y \in \mathbb{R}; |y - y_0| < \varepsilon\}$ if*

$$E[f(X)] = \int_{\mathbb{R}} f(y)p_{y_0}(y)dy$$

for any continuous and bounded function f whose support in $B_\varepsilon(y_0)$.

Our main purpose is to prove the local smoothness of the density of the solution of (1) under the following assumptions.

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Assumptions

There exists some $y_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that

$(A1)$: σ and b are bounded on the open ball $B_{6\varepsilon}(y_0)$. Moreover, $\inf_{x \in B_{6\varepsilon}(y_0)} |\sigma(x)| > \sigma_0 > 0$ for some constant σ_0.

$(A2)$ $\sigma \in C_b^\infty(B_{6\varepsilon}(y_0))$.

$(A3)$: $\sigma^{-1}b := \dfrac{b}{\sigma}$ is α-Hölder continuous on $B_{6\varepsilon}(y_0)$, where $\alpha \in (0,1)$.

To prove the local smoothness of the density, following lemmas are useful.

Lemma 1. *Let X be a $\mathbb{R}$-valued random variable and φ be its characteristic function. Assume that the following inequality holds for some positive constant C and $0 < \alpha < 1$.*

$$|\varphi(\theta)| \leq 1 \wedge (C|\theta|^{-(1+\alpha)}) \ (\forall \theta \in \mathbb{R}).$$

Then the density function of the law of X exists and is γ-Hölder continuous for any $0 < \gamma < \alpha$.

Lemma 2. *Let X be a $\mathbb{R}$-valued random variable, $\varepsilon > 0$ and ϕ_ε be an element of C_b^∞ which satisfies that*

$$1_{B_\varepsilon(0)} \leq \phi_\varepsilon \leq 1_{B_{2\varepsilon}(0)}.$$

Fix $y_0 \in \mathbb{R}$. Set $m_0 := E[\phi_\varepsilon(X - y_0)] > 0$ and consider $\mathcal{L}_{y_0}$ the probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f(y)\mathcal{L}_{y_0}(dy) = \frac{1}{m_0} E[f(X)\phi_\varepsilon(X - y_0)],$$

for all continuous and bounded function f. If $\mathcal{L}_{y_0}$ possesses a density $\tilde{p}_{y_0}$ then $p_{y_0} := m_0 \tilde{p}_{y_0}$ is the density function of X on $B_\varepsilon(y_0)$.

Thanks to Lemma 1 and Lemma 2, if for $t > 0$,

$$|E[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)]| \leq 1 \wedge (C|\theta|^{-(1+\gamma)}) \ (\forall |\theta| \geq 1) \tag{2}$$

holds for some positive constants C and γ, then for any $\gamma' \in (0, \gamma)$ the density function of the X_t exists and is γ'-Hölder continuous on $B_\varepsilon(y_0)$. Here, ϕ_ε is an element of $C_b^\infty(\mathbb{R})$ which satisfies the conditions of Lemma 2.

The main tool of our approach is Malliavin calculus which is well known as a method to prove the regularity of a solution of a SDE. However, in general the above solution X is not differentiable in Malliavin sense. To solve this problem, we use Girsanov's theorem and localize X by using some stopping times in order to deal with the local smoothness of the diffusion coefficient.

In our method, we consider a localization method for σ together with Girsanov's theorem in order to treat the regularity of the density. The localization allows to change the process X by a regularized version $\bar{X}$ for which Malliavin Calculus is applicable. The remaining problem is how to deal with the change of measure which contains the non-smooth function b. At this point, we use a similar argument as in [1], approximating the random variable $\bar{X}_t$ by a corresponding approximation. Then the change of measure is also approximated by its value at $t-\varepsilon$. This allows the use of the integration by parts formula. Finally, one needs to consider the approximation error which will finally lead to the following result.

Theorem 1. *Assume $(A1), (A2)$ and $(A3)$. Then for any initial value x_0, any $0 < t \leq T$ and any $0 < \gamma < \alpha$, the law of X_t has a γ-Hölder continuous density on $B_\varepsilon(y_0)$.*

For examples of applications of the results obtained here, see [1] and [3].

References

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