Density of stochastic differential equations driven by gamma processes

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November 12, 2011

Let a, $b > 0$ be constants, and fix $T > 0$. Let $J^i = \{J^i_t\,;\, t \in [0,T]\}$ $(1 \le i \le d)$ be (a,b)-gamma processes, that is, the process J^i is a one-sided pure-jump Lévy process without Gaussian component with the Lévy measure

$$\nu(dz) = g(z)dz, \quad g(z) = az^{-1}e^{-bz}, \quad z \in (0,+\infty),$$

and its characteristic function of the marginal for time $t \in [0,T]$ is

$$\mathbb{E}\big[e^{\sqrt{-1}\xi J^i_t}\big] = \big(1 - \sqrt{-1}\xi/b\big)^{-at}.$$

Suppose that the processes $J^1, \ldots, J^d$ are mutually independent. The marginal J^i_t at time $t \in [0,T]$ has the density function in closed form:

$$p^{J^i}_t(y) = b^{at}y^{at-1}e^{-by}/\Gamma(at), \quad y \in [0,+\infty).$$

Let $A_i \in C^\infty_b(\mathbb{R}^d\,;\,\mathbb{R}^d)$ $(0 \le i \le d)$ with the invertible condition:

$$\inf_{y\in\mathbb{R}^d} \inf_{z\in(0,+\infty)} |\det(I_d + \partial A_i(y)\,z)| > 0 \quad (1 \le i \le d).$$

For a non-random point $x \in \mathbb{R}^d$, we shall consider the $\mathbb{R}^d$-valued process $\{X_t\,;\, t \in [0,T]\}$ determined by the stochastic differential equation of the form:

$$dX_t = A_0(X_t)\,dt + A(X_{t-})\,dJ_t, \quad X_0 = x, \tag{1}$$

where $A = \big(A_1, \ldots, A_d\big)$ and $J_t = \big(J^1_t, \ldots, J^d_t\big)$. Then, there exists a unique solution $\{X_t\,;\, t \in [0,T]\}$ to (1) such that, for each $t \in [0,T]$, the function $\mathbb{R}^d \ni x \longmapsto X_t \in \mathbb{R}^d$ has a C^∞-modification, and its Jacobi matrix is invertible a.s. In this talk, we shall focus on the sensitivity, and the error estimate on the densities between the solution and the driving gamma process. This is based upon joint work with Vlad Bally (Université Paris-Est Marne-la-Vallée, France).

Let $C_1 > 0$ be a constant, and $\Xi \in C^\infty_b(\mathbb{R}^d \otimes \mathbb{R}^d\,;\,[0,1])$ such that

$$\Xi(B) = 0 \ (0 \le |\det B| \le C_1/2), \quad \Xi(B) = 1 \ (|\det B| \ge C_1).$$

The Girsanov transform leads to get the integration by parts formula for X_T.

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Theorem 1 *For $\varphi \in C_b^1(\mathbb{R}^d\,;\,\mathbb{R})$, the following equality holds:*

$$\mathbb{E}\big[\partial_k\varphi(X_T)\,\Xi(V_T^X)\big] = \mathbb{E}\big[\varphi(X_T)\,\Theta_k(X_T,\Xi(V_T^X))\big]$$

for $1 \leq k \leq d$, where V_T^X is the Malliavin covariance matrix for X_T.

Suppose the uniformly elliptic condition on A_i $(1 \leq i \leq d)$:

$$\inf_{\zeta\in\mathbb{S}^{d-1}}\ \inf_{y\in\mathbb{R}^d} \zeta\cdot A(y)\,A(y)^* y \geq C_2,$$

under which there exists a C^∞-density for X_T with respect to the Lebesgue measure on $\mathbb{R}^d$ via Theorem 1. Using Theorem 1 also enables us to see that

Theorem 2 *It holds that*

$$\frac{\mathbb{E}\big[\mathbb{I}_{(X_T\in dy)}\,\Xi(V_T^X)\big]}{dy} = \mathbb{E}\left[\sum_{k=1}^d \partial_k Q_d(X_T-y)\,\Theta_k(X_T,\Xi(V_T^X))\right],$$

where Q_d is the fundamental solution to the equation $\Delta Q_d = \delta_0$.

We shall rewrite the equation (1) as follows:

$$\begin{aligned} X_T &= \big\{x + A(x)J_T\big\} + \Big\{\int_0^T A_0(X_s)ds + \int_0^T \big(A(X_{s-}) - A(x)\big)dJ_s\Big\} \\ &=: G_T + R_T. \end{aligned}$$

Let $C_3 > 0$ be a constant, and $\psi_{1,i} \in C_b^\infty([0,+\infty)\,;\,[0,1])$ $(1 \leq i \leq d)$ with

$$\psi_{1,i}(u_i) = 1\ (u_i \geq C_3), \quad \psi_{1,i}(u_i) = 0\ (u_i \leq C_3/2).$$

Define $\psi_1(u) = \prod_{i=1}^d \psi_{1,i}(u_i)$ and $p_T^J(u) = \prod_{i=1}^d p_T^{J^i}(u_i)$ for $u = (u_1,\ldots,u_d)$.

Theorem 3 *It holds that*

$$p_T^X(y) \geq \tilde{p}_T^G(y) - \mathcal{E}_T,$$

where

$$\begin{aligned} \tilde{p}_T^G(y) &= \psi_1\big(A(x)^{-1}(y-x)\big)p_T^J\big(A(x)^{-1}(y-x)\big), \\ \mathcal{E}_T &= C_4\big(|R_T|_p + \|V_T^{\tilde{R}}\|_p + \|H_T^{\tilde{R}}\|_p\big). \end{aligned}$$

References

[1] V. Bally and L. Caramellino: Riesz transform and integration by parts formulas for random variables, *Stochastic Process Appl.*, **121**, 1332–1355 (2011).

[2] V. Bally and A. Takeuchi: Lower bounds for densities of stochastic differential equations driven by gamma processes, preprint.

[3] J. M. Bismut: Calcul des variations stochastique et processus de sauts, *Z. Wharsch. verw. Gebiete*, **63**, 147–235 (1983).

[4] R. Kawai and A. Takeuchi: Greeks formulas for an asset price model with gamma processes, *Math. Finance*, **21**, 723–742 (2011).