Bakry-Émery 型微分評価と関連する話題

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1. Bakry-Émery の微分評価とは

P_t: 生成作用素 $\mathcal{L}$ を持つ半群 (P_t "=" $\mathrm{e}^{t\mathcal{L}}$)

Bakry-Émery の (L^p-) 微分評価:

$$|\nabla P_t f|(x) \le \mathrm{e}^{-Kt} P_t(|\nabla f|^p)(x)^{1/p} \quad (G_p)$$

$$(p \in [1, \infty),\ K \in \mathbb{R})$$

[D. Bakry & M. Émery, '84]

- P_t の定義される関数空間: 標準的には,
  P_t: 強連続縮小, 対称, Markov 的 on $L^2(\mu)$
  $\left(\Rightarrow \exists \text{symm. contr. extension on } L^p(\mu) \quad (1 \le p \le \infty) \right)$
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★ もし P_t: Markovian
(i.e. $0 \le f \le 1 \Rightarrow 0 \le P_t f \le 1$) なら,
$p' < p$ のとき「$(G_{p'}) \Rightarrow (G_p)$」
($\because$ Hölder ineq. for $(f, g) \mapsto P_t(fg)$)

__例__ ($\mathbb{R}^m$ の熱半群, i.e. $\mathcal{L} = \Delta$)

$f : \mathbb{R}^m \to \mathbb{R}$

$$P_t f(x) := \int_{\mathbb{R}^m} p_t(x, y) f(y) dy,$$

$$p_t(x, y) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x - y|^2}{4t} \right)$$

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(∵ (可換な) 変数変換 (平行移動))

$$\Rightarrow |\nabla P_t f|(x) \leq P_t(|\nabla f|)(x),$$

i.e. (G_1) with $K = 0$

微分 $|\nabla f|$ について:

★ 平方場作用素 (carré du champ)

$$\Gamma(f,g) := \frac{1}{2}\left(\mathfrak{L}(fg) - g\mathfrak{L}f - f\mathfrak{L}g\right)$$

(例：$\mathfrak{L} = \Delta$ on $\mathbb{R}^m \Rightarrow \Gamma(f,f) = |\nabla f|^2$)

- $$\Gamma(f,f) = \lim_{t\to 0}\frac{1}{t}\left(P_t(f^2) - (P_t f)^2\right) \geq 0$$
- $P_t \leftrightarrow (\mathcal{E},\mathcal{F})$: Dirichlet form on $L^2(\mu)$ なら,

$$\int h\Gamma(f,f)d\mu = \frac{1}{2}\left(2\mathcal{E}(hf,f) - \mathcal{E}(h,f^2)\right)$$

$$(h, f \in \mathcal{F} \cap L^\infty)$$

以下，この § では，以下を仮定：

- $\exists\Gamma$
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$\Rightarrow$ 例えば $(G_{\mathbf{2}})$ は，

$$\boldsymbol{\Gamma}(P_t f, P_t f) \le \mathrm{e}^{-2Kt} P_t(\boldsymbol{\Gamma}(f,f))$$

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★ 「“$=$” at $t=0$」を利用して $t=0$ で微分：

$$2\Gamma(f, \mathcal{L}f) \leq -2K\Gamma(f,f) + \mathcal{L}(\Gamma(f,f))$$

$$2\Gamma(f,\mathfrak{L}f) \leq -2K\Gamma(f,f) + \mathfrak{L}(\Gamma(f,f))$$

$$\Downarrow \Gamma_2(f,f) := \frac{1}{2}\left(\mathfrak{L}(\Gamma(f,f)) - 2\Gamma(f,\mathfrak{L}f)\right)$$

Γ_2-条件:

$$\Gamma_2(f,f) \geq K\Gamma(f,f) \qquad (\Gamma_2(K))$$

命題

$$(G_2) \Leftrightarrow (\Gamma_2(K))$$

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"証明"

「$(\Gamma_2(K)) \Rightarrow (G_2)$」のみ示せば良い.

$$\Phi(s) := P_s(\Gamma(P_{t-s}f, P_{t-s}f))$$

- $\Phi(0) = \Gamma(P_t f, P_t f)$, $\Phi(t) = P_t(\Gamma(f, f))$
- $(\Gamma_2(K)) \Rightarrow \Phi'(s) \geq 2K\Phi(s)$
  $\Rightarrow \Phi(0) \leq \mathrm{e}^{-2Kt}\Phi(t)$ ■

つまり，時間変数 t に関して，

微分

$$(G_2) \quad \overset{\text{微分}}{\Longrightarrow} \quad (\Gamma_2(K))$$

$$(G_2) \quad \underset{\text{積分}}{\Longleftarrow} \quad (\Gamma_2(K))$$

積分

注

- (G_2) の weak formulation:
$$\int g\Gamma(P_t f, P_t f)\,d\mu \le \mathrm{e}^{-2Kt} \int g P_t(\Gamma(f,f))\,d\mu$$
$(\forall g \ge 0$: "nice" test function)
に依れば，弱い仮定で厳密に証明可能
[cf. Gigli-K.-Ohta '10, Theorem 4.8]

- この議論に，Γ や $\mathcal{L}$ の derivation property
(or $(\mathcal{E}, \mathcal{F})$ の strong locality) は不要

Γ_2 の計算例

(i) $V : \mathbb{R}^m \to \mathbb{R}$, $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ on $\mathbb{R}^m$

$$\Gamma(f, f) = |\nabla f|^2,$$

$$\Gamma_2(f, f) = \| \mathrm{Hess}\, f \|_{\mathrm{HS}} + \mathrm{Hess}\, V (\nabla f, \nabla f)$$

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特に，Ornstein-Uhlenbeck 過程の場合：

$$\mathcal{L} = \Delta - Kx \cdot \nabla \ (\leftrightarrow V(x) = \frac{K}{2}|x|^2)$$

$$\Rightarrow (\Gamma_2(K))$$

(ii) $\mathcal{L} = \Delta$ on a Riemannian manifold

$$\Gamma_2(f,f) = \frac{1}{2}\left(\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f\rangle\right)$$
$$= \|\mathrm{Hess}\, f\|_{\mathrm{HS}} + \mathrm{Ric}(\nabla f, \nabla f)$$

(Bochner-Weitzenböck の公式)

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★ 微分幾何で「$\mathrm{Ric} \geq K$」から従う事の多くは，$(\Gamma_2(K))$ からも (抽象的な枠組で) 従う

(G_2) の応用

- Poincaré inequality (for P_t)

$$P_t(f^2) - (P_t f)^2 \le \frac{1 - e^{-2Kt}}{K} P_t(\Gamma(f, f))$$

- Reversed Poincaré inequality (for P_t)

$$P_t(f^2) - (P_t f)^2 \ge \frac{e^{2Kt} - 1}{K} \Gamma(P_t f, P_t f)$$

$$\left(\Rightarrow |\nabla P_t f| \le \sqrt{\frac{K}{e^{2Kt} - 1}} \operatorname{osc} f \right)$$

証明

$$\Psi(s) := P_s((P_{t-s}f)^2)$$

- $\Psi(0) = (P_t f)^2$, $\Psi(t) = P_t(f^2)$
- $\Psi'(s) = 2P_s(\Gamma(P_{t-s}f, P_{t-s}f))$
  By (G_2),
  † $\Psi'(s) \leq 2\mathrm{e}^{-2K(t-s)}P_s P_{t-s}(\Gamma(f,f))$
  † $\Psi'(s) \geq 2\mathrm{e}^{2Ks}\Gamma(P_s P_{t-s}f, P_s P_{t-s}f)$

$\Rightarrow$ conclusion by integration in s ■

注

- 形式的には,

$$\text{Poincaré or rev. Poincaré ineq.} \Rightarrow (\Gamma_2(K))$$

(両辺を $t = 0$ で 2 次まで Taylor 展開)

- これらの議論には, Γ および $\mathcal{L}$ の derivation property は不要

条件 (G_1) について

- **強い仮定** のもと, $(\Gamma_2(K)) \Rightarrow (G_1)$

  ↰ $\exists \mathcal{C} \subset \mathrm{Dom}(\mathcal{L})$ dense s.t.

  - $\mathcal{C}$ は積で閉じている線形空間
  - $P_t\mathcal{C} \subset \mathcal{C}$, $\mathcal{L}\mathcal{C} \subset \mathcal{C}$
  - $\forall \varphi$: $C^\infty(\mathbb{R})$, $f \in \mathcal{C} \Rightarrow \varphi(f) \in \mathcal{C}$

- $(G_1) \Rightarrow$ log-Sobolev ineq. (for P_t):

$$P_t(f^2 \log f^2) - P_t(f^2) \log(P_t(f^2)) \le \frac{2(1 - \mathrm{e}^{-2Kt})}{K} P_t(\Gamma(f, f))$$

Derivation property

$\varphi \in C^2(\mathbb{R})$, $\varphi(0) = 0$

- $f \in \mathrm{Dom}(\mathfrak{L})$

  $\Rightarrow \varphi(f) \in \mathrm{Dom}(f)$,

  $$\mathfrak{L}(\varphi(f)) = \varphi'(f)\mathfrak{L}f + \varphi''(f)\Gamma(f, f)$$

- $f \in \mathrm{Dom}(\Gamma)$

  $\Rightarrow \varphi(f) \in \mathrm{Dom}(\Gamma)$,

  $$\Gamma(\varphi(f), g) = \varphi'(f)\Gamma(f, g)$$

Proof of 「$(G_1) \Rightarrow$ log-Sobolev」

(P_t: Markov は仮定)

$$\boxed{\Psi(s) := P_{s}(\varphi(P_{t-s}f))} \qquad (\varphi \in C^2(\mathbb{R}))$$

$\Rightarrow$ By the derivation property,

$$\Psi'(s) = P_s(\varphi''(P_{s-t}f)\Gamma(P_{s-t}f, P_{s-t}f))$$

- $\varphi(u) := u\log u \Rightarrow \varphi''(u) = \dfrac{1}{u}$

  $\Rightarrow$ <u>Goal</u>: For $f \geq 0$,

$$\Psi'(s) \leq 4\mathrm{e}^{-2K(t-s)}P_t(\Gamma(\sqrt{f}, \sqrt{f}))$$

By (G_1) & Schwarz ineq. for $(g,h) \mapsto P_{t-s}(gh)$,

$$\Psi'(s) = P_s\left(\frac{\Gamma(P_{t-s}f, P_{t-s}f)}{P_{t-s}f}\right)$$

$$\leq \mathrm{e}^{-2K(t-s)} P_s\left(\frac{\left(P_{t-s}(\sqrt{\Gamma(f,f)})\right)^2}{P_{t-s}f}\right)$$

$$\leq \mathrm{e}^{-2K(t-s)} P_s P_{t-s}\left(\sqrt{\frac{\Gamma(f,f)}{f}}^{\,2}\right)$$

$$= 4\mathrm{e}^{-2K(t-s)} P_t\left(\Gamma(\sqrt{f}, \sqrt{f})\right) \quad \blacksquare$$

大域的 Poincaré/log-Sobolev

もし, $K > 0$ & $\lim_{t\to\infty} P_t f \equiv \int f d\mu$ ならば,

$$P_t(f^2) - (P_t f)^2 \leq \frac{1 - \mathrm{e}^{-2Kt}}{K} P_t(\Gamma(f, f))$$

$$\Downarrow$$

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \frac{1}{K} \int \Gamma(f, f) d\mu$$

大域的 Poincaré/log-Sobolev

もし，$K > 0$ & $\lim_{t\to\infty} P_t f \equiv \int f d\mu$ ならば，

$$P_t(f^2 \log f^2) - P_t(f^2) \log P_t(f^2)$$

$$\leq \frac{2(1-\mathrm{e}^{-2Kt})}{K} P_t(\Gamma(f,f))$$

$$\Downarrow$$

$$\int f^2 \log f^2 d\mu - \int f^2 d\mu \log\left(\int f^2 d\mu\right)$$

$$\leq \frac{2}{K} \int \Gamma(f,f) d\mu$$

(G_1) への確率論的アプローチ $\cdots$ 結合法

例

$\mathcal{L} := \Delta - \nabla V \cdot \nabla$ on $\mathbb{R}^m$

$$\updownarrow$$

$$dX_t^x = \sqrt{2}dW_t - \nabla V(X_t^x)dt, \quad X_0^x = x$$

仮定: $\forall x, y \in \mathbb{R}^m$,

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq K|x - y|^2$$

$$(\Uparrow \mathrm{Hess}\, V \geq K)$$

X_t^x, X_t^y: 前掲の SDE の (共通の W_t に対する) 解

$$\Downarrow$$

$$d(X_t^x - X_t^y) = -\left(\nabla V(X_t^x) - \nabla V(X_t^y)\right) dt$$

$$\Downarrow$$

$$d|X_t^x - X_t^y|^2 \le -2K|X_t^x - X_t^y|^2 dt$$

$$\Downarrow$$

$$|X_t^x - X_t^y|^2 \le \mathrm{e}^{-2Kt}|x - y|^2$$

$$\Downarrow$$

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= |\mathbb{E}[f(X_t^x) - f(X_t^y)]| \\
&\leq \mathbb{E}[|\nabla f|(X_t^x)|X_t^x - X_t^y| + o(|X_t^x - X_t^y|)] \\
&\leq \mathrm{e}^{-Kt}\mathbb{E}[|\nabla f|(X_t^x)]|x - y| + o(|x - y|)
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
|\nabla P_t f|(x) &= \limsup_{y \to x} \frac{|P_t f(x) - P_t f(y)|}{|x - y|} \\
&\leq \mathrm{e}^{-Kt}\mathbb{E}[|\nabla f|(X_t^x)] \\
&= \mathrm{e}^{-Kt} P_t(|\nabla f|)(x)
\end{aligned}$$

注

- 結合法の方法は，Riemann 多様体上や無限次元空間上にも拡張されている.
- 設定によっては，解析は単純ではない(拡散係数が定数でない場合の SDE 等).
- それらの拡張では，Bakry-Émery の議論「$(\Gamma_2(K)) \Rightarrow (G_1)$」で用いた，強い仮定は要らない.

2. Wasserstein 距離の Lipschitz 型評価との双対性

(X,d): ポーランド空間, d: 測地距離

- $\tilde{d}$: X 上の連続な測地距離関数
  (例: $\tilde{d} = \mathrm{e}^{-Kt} d$)

- $(P(x,\cdot))_{x\in X} \subset \mathcal{P}(X)$: Markov 核

$$Pf(x) := \int_X f(y)\, dP(x,dy),$$

$$P^*\mu(A) := \int_X P(x,A)\mu(dx)$$

(例: $P(x,dy) = p_t(x,dy)$ 推移確率)

L^p-Wasserstein 距離

For $p \in [1, \infty]$, $\mu, \nu \in \mathcal{P}(X)$

$$W_{d,p}(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0, \infty],$$

$\Pi(\mu, \nu) \subset \mathcal{P}(X \times X)$,

$$\Pi(\mu, \nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

((μ, ν) のカップリング全体)

“微分” $|\nabla f|$

$$|\nabla_d f|(x) := \limsup_{y \to x} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

- $|\nabla_d f|(x)$: x での d-局所 Lipschitz 定数
- $\|\nabla_d f\|_\infty$: d-Lipschitz 定数, i.e.

$$|f(x) - f(y)| \le \|\nabla_d f\|_\infty d(x, y)$$

$$W_{d,q}(P^*\mu, P^*\nu) \le W_{\tilde{d},q}(\mu,\nu) \qquad (W_q)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^p)(x)^{1/p} \qquad (G'_p)$$

$$(\mu,\nu \in \mathcal{P}(X),\ f \in C_b^{\mathrm{Lip}}(X))$$

定理[cf. K. '10]

For $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(W_q) \Leftrightarrow (G'_p)$$

注

- For $p' < p$,
  $(G'_{p'}) \Rightarrow (G'_{p})$ and $(W_{q'}) \Rightarrow (W_{q})$
- $(W_1) \Leftrightarrow (G'_\infty)$ は既知
  (Kantorovich-Rubinstein formula による)
- $(W_\infty) \Rightarrow (G'_1)$ は (本質的に) 既知
  (結合法)
- [K. '10] で課していた，技術的な仮定は不要

証明の概略

(for $p, q \in (1, \infty)$)

[A] $(W_q) \Rightarrow (G'_p)$:

$$W_{d,q}(P^*\mu, P^*\nu) \le W_{\tilde{d},q}(\mu,\nu) \qquad (W_q)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^p)(x)^{1/p} \qquad (G'_p)$$

Goal:

$$|Pf(x) - Pf(y)|$$

$$\le P(|\nabla_d f|^p)(x)^{1/p}\tilde{d}(x,y) + o(\tilde{d}(x,y))$$

π: $W_{d,p}(P^*\delta_x, P^*\delta_y)$ の minimizer とする

$$
\begin{aligned}
&\boldsymbol{Pf(x) - Pf(y)} \\
&\quad = \int_{\boldsymbol{X}} \boldsymbol{f dP^{*} \delta_{x}} - \int_{\boldsymbol{X}} \boldsymbol{f dP^{*} \delta_{y}} \\
&\quad = \int_{\boldsymbol{X \times X}} \boldsymbol{(f(z) - f(w))\pi(dzdw)} \\
&\quad = \int_{\boldsymbol{X \times X}} \boldsymbol{1_{\{d \leq r\}}}(\underline{\qquad\qquad\qquad})\boldsymbol{\pi(dzdw)} \\
&\qquad + \int_{\boldsymbol{X \times X}} \boldsymbol{1_{\{d > r\}}}(\underline{\qquad\qquad\qquad})\boldsymbol{\pi(dzdw)} \\
&\quad = \boldsymbol{(I) + (II)}
\end{aligned}
$$

$$G_r f(z) := \sup_{d(w,z)\le r} \frac{|f(z) - f(w)|}{d(z,w)}$$

$$\Downarrow$$

$$(I) \le \int_{X\times X} G_r f(z) d(z,w)\pi(dzdw)$$

$$\le P(|G_r f|^p)(x)^{1/p} W_{d,q}(P^*\delta_x, P^*\delta_y)$$

$$(W_q) \quad \boxed{\le} P(|G_r f|^p)(x)^{1/p} \tilde{d}(x,y)$$

$$= P(|\nabla_d f|^p)(x)^{1/p} \tilde{d}(x,y) + o(\tilde{d}(x,y))$$

$$\text{if } r = o(1) \text{ as } \tilde{d}(x,y) \to 0$$

$$
\begin{aligned}
(II) &\leq 2\|f\|_\infty \pi(d > r) \\
&\leq \frac{1}{r^q} W_{d,q}(P^*\delta_x, P^*\delta_y)^q \\
&\leq \left(\frac{\tilde{d}(x,y)}{r}\right)^q = o(\tilde{d}(x,y))
\end{aligned}
$$

for a suitable choice of r

(i.e. $\tilde{d}(x,y)^{1/p} = o(r)$) //

[B] $(G'_p) \Rightarrow (W_q)$:

$$|\nabla_{\tilde{d}} Pf|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \qquad (G'_p)$$

$$W_{d,q}(P^*\mu, P^*\nu) \leq W_{\tilde{d},q}(\mu, \nu) \qquad (W_q)$$

[B] $(G_p') \Rightarrow (W_q)$:

$$|\nabla_{\tilde{d}} Pf|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \qquad (G_p')$$

$$W_{d,q}(P^*\mu, P^*\nu) \leq W_{\tilde{d},q}(\mu, \nu) \qquad (W_q)$$

- (W_q) を $\mu = \delta_x, \nu = \delta_y$ の場合のみ示す

$\Rightarrow$ Goal: $\frac{1}{q} W_{d,q}(P^*\delta_x, P^*\delta_y)^q \leq \frac{1}{q} \tilde{d}(x, y)^q$

Kantorovich duality

$$\frac{W_{d,q}(\mu,\nu)^q}{q} = \sup_{f \in C_b^{\mathrm{Lip}}(X)} \left[\int_X \hat{f} d\mu - \int_X f \, d\nu \right],$$

$$\hat{f}(x) := \inf_{y \in X} \left[f(y) + \frac{d(x,y)^q}{q} \right]$$

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★ $\frac{1}{q} W_{d,q}(P^* \delta_x, P^* \delta_y)^q$

$$= \sup_f \left[P\hat{f}(x) - Pf(y) \right]$$

Kantorovich duality

$$\frac{W_{d,q}(\mu,\nu)^q}{q} = \sup_{f \in C_b^{\mathrm{Lip}}(X)} \left[\int_X \hat{f}\, d\mu - \int_X f\, d\nu \right],$$

$$\hat{f}(x) := \inf_{y \in X} \left[f(y) + \frac{d(x,y)^q}{q} \right]$$

★ $$\frac{1}{q} W_{d,q}(P^*\delta_x, P^*\delta_y)^q = \sup_f \left[P\hat{f}(x) - Pf(y) \right]$$

$\Rightarrow$ **Goal:** $$\sup_f \left[P\hat{f}(x) - Pf(y) \right] \le \frac{\tilde{d}(x,y)^q}{q}$$

Kantorovich duality の "$\geq$" の証明

$$\pi \in \Pi(\mu, \nu),$$

$$g, f \in C_b^{\mathrm{Lip}}(X), \quad \boxed{g(x) - f(y) \leq \frac{d(x,y)^q}{q}}$$

$$\Downarrow$$

$$\int_X g \, d\mu - \int_X f \, d\nu \leq \frac{1}{q} \|d\|_{L^q(\pi)}^q$$

$\rightsquigarrow$ maximize $\qquad\qquad$ $\rightsquigarrow$ minimize

$$\sup_f \left[\int_X \hat{f} d\mu - \int_X f d\nu \right] \leq \frac{W_{d,q}(\mu,\nu)^q}{q} \quad //$$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x,y)^q}{q t^{q-1}} \right],$$

$$Q_0 f(x) := f(x)$$

★ $Q_1 f = \hat{f}$

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Goal: $\sup_f \left[P\hat{f}(x) - Pf(y) \right] \leq \frac{\tilde{d}(x,y)^q}{q}$

- $\tilde{\gamma} : [0,1] \to X$: $\tilde{d}$-測地線, $\tilde{\gamma}_0 = y$, $\tilde{\gamma}_1 := x$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y\in X}\left[f(y) + \frac{d(x,y)^q}{q t^{q-1}}\right],$$

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$\Rightarrow$ **Goal**: $\forall f \in C_b^{\mathrm{Lip}}(X)$,

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0) \le \frac{\tilde{d}(x,y)^q}{q}$$

- $\tilde{\gamma} : [0,1] \to X$: $\tilde{d}$-測地線, $\tilde{\gamma}_0 = y$, $\tilde{\gamma}_1 := x$

Key idea:

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0)$$
$$= \int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt$$

と変形 (補間)

Key idea:

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0) = \int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt$$

と変形 (補間)

実現には,

$$t \mapsto PQ_t f(\tilde{\gamma}_t): \text{ Lipschitz}$$

を示せば充分

補題

For $f \in C_b^{\mathrm{Lip}}(X)$,
$(t, z) \mapsto PQ_t f(z)$: Lipschitz w.r.t $|\cdot|$ & $\tilde{d}$

証明

[Balogh & Engoulatov & Hunziker & Maasalo '09]:

$$|Q_t f(z) - Q_t f(w)| \le \|\nabla f\|_\infty d(z, w),$$

$$|Q_t f(z) - Q_s f(z)| \le \frac{\|\nabla f\|_\infty^p}{p} |t - s|$$

$\therefore (G'_p) \Rightarrow$ conclusion ■

$$\Downarrow$$

Goal: For $\forall f \in C_b^{\mathrm{Lip}}(X)$,

$$\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t))dt \le \frac{\tilde{d}(x,y)^q}{q}$$

★ 「$\partial_t (PQ_t f(\tilde{\gamma}_t)) \le \dfrac{\tilde{d}(x,y)^q}{q}$ for a.e. t 」

を示せば充分

By [Ambrosio & Gigli & Savaré '08, Lemma 4.3.4],

$$
\begin{aligned}
\partial_t PQ_t f(\tilde{\gamma}_t) &\le \overline{\lim_{h\downarrow 0}} \frac{PQ_{t+h} f(\tilde{\gamma}_t) - PQ_t f(\tilde{\gamma}_t)}{h} \\
&\quad + \overline{\lim_{h\downarrow 0}} \frac{PQ_t f(\tilde{\gamma}_t) - PQ_t f(\tilde{\gamma}_{t-h})}{h} \\
&= (I) + (II)
\end{aligned}
$$

(Leibniz 則)

By the def. of $|\nabla_{\tilde{d}} \cdot |$,

$$(II) = \overline{\lim_{h\downarrow 0}} \frac{PQ_t f(\tilde{\gamma}_t) - PQ_t f(\tilde{\gamma}_{t-h})}{h}$$

$$\leq \tilde{d}(x,y) |\nabla_{\tilde{d}} PQ_t f| (\tilde{\gamma}_t)$$

$$(G'_p) \boxed{\leq} \tilde{d}(x,y) P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)^{1/p}$$

($\tilde{d}(x,y)$: speed of $\tilde{\gamma}$)

★ By [Ambrosio & Gigli & Savaré '11],

For $\forall t > 0$, $\forall z \in X$, $\exists \partial_t^+ Q_t f(z)$,

$$\partial_t^+ Q_t f(z) + \frac{1}{p} |\nabla_d Q_t f|(z)^p \leq 0 \quad (*)$$

(∗) が正しい (intuitive な) 理由

「$\exists y_0$ s.t. $Q_s f(z) = f(y_0) + \dfrac{d(z, y_0)^q}{q s^{q-1}}$

for $\forall s \approx t$, $\forall z \approx x$」と仮定

$$\Downarrow$$

$$\partial_t Q_t f(x) = -\frac{q-1}{q}\left(\frac{d(x, y_0)}{t}\right)^q$$

$$\frac{1}{p}|\nabla_d Q_t f|^p = \frac{1}{p}\left(\frac{d(x, y_0)}{t}\right)^{p(q-1)}$$

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For $\forall t > 0$, $\forall z \in X$, $\exists \partial_t^+ Q_t f(z)$,

$$\partial_t^+ Q_t f(z) + \frac{1}{p} |\nabla_d Q_t f|(z)^p \leq 0 \quad (*)$$

$$\Downarrow$$

$$(I) = \overline{\lim_{h \downarrow 0}} \frac{PQ_{t+h} f(\tilde{\gamma}_t) - PQ_t f(\tilde{\gamma}_t)}{h}$$

Fatou $\boxed{\leq} P(\partial_t^+ Q_t f)(\tilde{\gamma}_t)$

$(*)$ $\boxed{\leq} -\frac{1}{p} P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)$

$$(I) \le \tilde{d}(x,y) P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)^{1/p}$$

$$(II) \le -\frac{1}{p} P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)$$

$$\Downarrow$$

$\sigma := P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)^{1/p}$ とおくと,

$$\partial_t (PQ_t f(\tilde{\gamma}_t)) \le \tilde{d}(x,y)\sigma - \frac{1}{p}\sigma^p$$

$$\le \sup_{u \ge 0} \left[\tilde{d}(x,y) u - \frac{1}{p} u^p \right] = \frac{\tilde{d}(x,y)^q}{q} \quad \blacksquare$$

証明の概観

$$\frac{W_{d,q}(P^*\delta_x, P^*\delta_y)^q}{q}$$

Kant. duality $\boxed{=} \sup_f [PQ_1 f(x) - PQ_0 f(y)]$

補間 $\boxed{=} \sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t))dt\right]$

Leibniz 則,

(G_p'), $(*)$ $\boxed{\leq} \dfrac{\tilde{d}(x,y)^q}{q}$

まとめると，**空間変数**に関して，

微分

$$(W_q) \quad \overset{\Rightarrow}{\underset{\Leftarrow}{}} \quad (G'_p)$$

積分

注

$(G'_p) \Rightarrow (W_q)$ の証明では，以下を要請:

(G'_p) が $\forall x \in X$ (or "on a. e. curves") で成立.

しかし，関数解析的手法では,「(G_p) a.e.」が普通.

一方，以下が成立すれば (G'_p) は v-a.e. で充分;

(i) $W_{d,p}$ が $\mathcal{P}_p(X)$ 上の測地距離

(ii) $\mu_0, \mu_1 \ll v \Rightarrow \forall$ geod. $(\mu_t)_{t\in[0,1]}$, $\mu_t \ll v$

($p = 2$ & MCP ／ Sturm-Lott-Villani 条件下で該当)

[cf. Gigli-K.-Ohta '10, Theorem 4.8(ii) $\Rightarrow$ (i)]

応用

[A] 直積不変性

(X_i, d_i): ポーランド空間, d_i: 測地距離,

$\tilde{d}_i$: 連続な測地距離関数,

P_i: X_i 上の Markov 核,

$$W_{d_i,2}(P_i^*\mu, P_i^*\nu) \le W_{\tilde{d}_i,2}(\mu,\nu)$$

$$\Downarrow$$

$P := P_1 \times P_2$: $X_1 \times X_2$ 上の Markov 核,

$d := \sqrt{d_1^2 + d_2^2}$: 直積距離 ($\tilde{d}$ も同様) で,

$$W_{d,2}(P^*\mu, P^*\nu) \le W_{\tilde{d},2}(\mu,\nu)$$

証明

$\boldsymbol{x^{(j)} = (x_1^{(j)}, x_2^{(j)}) \in X \times X}$,

$\boldsymbol{\mu = \delta_{x^{(1)}}}$, $\boldsymbol{\nu = \delta_{x^{(2)}}}$ で示せば充分

- $\boldsymbol{\pi_i}$: minimizer of $\boldsymbol{W_{d_i,2}(P^*\delta_{x_i^{(1)}}, P^*\delta_{x_i^{(2)}})}$

$$\Downarrow$$

$$\boldsymbol{\pi_1 \times \pi_2 \in \Pi(P^*\delta_{x^{(1)}}, P^*\delta_{x^{(2)}})}$$

$$
\begin{aligned}
W_{d,2}(P^*\delta_{x^{(1)}}, P^*\delta_{x^{(2)}})^2
&\leq \int_{X_1 \times X_1 \times X_2 \times X_2} d^2 \, d(\pi_1 \times \pi_2) \\
&= \|d_1\|_{L^2(\pi_1)}^2 + \|d_2\|_{L^2(\pi_2)}^2 \\
\text{仮定} \quad &\boxed{\leq} \, d_1(x_1^{(1)}, x_1^{(2)})^2 + d_2(x_2^{(1)}, x_2^{(2)})^2 \\
&= d(x^{(1)}, x^{(2)})^2 \quad \blacksquare
\end{aligned}
$$

[B] 従属操作

ξ_t: subordinator, i.e. $\xi_t \in \mathcal{P}([0,\infty))$,

$$\int_0^\infty \mathrm{e}^{-\lambda s}\xi_t(ds) = \mathrm{e}^{-t\phi(\lambda)}$$

P_t: X 上の推移確率族, $K \geq 0$,

$$W_{d,q}(P_t^*\mu, P_t^*\nu) \leq \mathrm{e}^{-Kt} W_{d,q}(\mu,\nu)$$

と仮定．このとき $\hat{P}_t = \int_0^\infty P_s \xi_t(ds)$ について,

$$W_{d,q}(\hat{P}_t^*\mu, \hat{P}_t^*\nu) \leq \mathrm{e}^{-\phi(qK)t/q} W_{d,q}(\mu,\nu)$$

証明

$\pi_t \in \Pi(P_t^* \mu, P_t^* \nu)$: min. of $W_{d,q}(P_t^* \mu, P_t^* \nu)$

$$\Downarrow$$

$$\int_0^\infty \pi_s \xi_t(ds) \in \Pi(\hat{P}_t^* \mu, \hat{P}_t^* \nu)$$

$$
\begin{aligned}
\boldsymbol{W}_{d,q}(\hat{P}_t^*\mu, \hat{P}_t^*\nu)^q
&\leq \int_0^\infty \int_{\boldsymbol{X}\times\boldsymbol{X}} \mathrm{d}(z,w)^q \pi_s(dz\,dw)\xi_t(ds) \\
&\leq \int_0^\infty \mathrm{e}^{-qKs} \boldsymbol{W}_{d,q}(\mu,\nu)^q \xi_t(ds) \\
\text{仮定} \quad &= \mathrm{e}^{-\phi(qK)t} \boldsymbol{W}_{d,q}(\mu,\nu)^q \quad \blacksquare
\end{aligned}
$$

[C] 熱半群での，(W_q) or (G'_p) の導出例

- Lie 群上, Hörmander 型作用素が生成する半群
  - (G'_p) $(\forall p > 1)$ or (G'_1)
    ((G_p) と異質な評価／解析的手法で導出)

$\Rightarrow$ (W_q) $(1 \leq \forall q < \infty)$ or (W_∞) [K. '10]

- (コンパクト)Alexandrov 空間
  - (W_2), $\tilde{d} = \mathrm{e}^{-Kt} d$
    (特異空間での評価／最適輸送理論と融合)

$\Rightarrow$ (G_2) [Gigli-K.-Ohta '10]

3. 関連する話題・課題

(1) “微分” について

- Bakry-Émery の元々の議論 (と結果)

  $\leftsquigarrow |\nabla f| = \sqrt{\Gamma(f,f)}$

- (W_q) との双対性

  $\leftsquigarrow |\nabla f|$: 局所 Lipschitz 定数

Q.

d が intrinsic distance, i.e.

$$d(x,y) = \sup_f \{f(x) - f(y); \Gamma(f,f) \le 1 \text{ a.e.}\}$$

ならば，2 つの微分は一致するか？

A. [Koskela & Zhou '11]

- 一般には, NO.
- (1,2)-local Poincaré ineq. および volume doubling property の下,

  [Newtonnian property] $\Leftrightarrow$ YES.

(2) $(\boldsymbol{G_p}) \Rightarrow (\boldsymbol{G_{p'}})$ for $p > p'$

- $p' < p$ のとき，$(G_{p'}) \Rightarrow (G_p)$

Q.

逆はいつ成り立つか？

A.

- Bakry-Émery's argument:
  $(G_2) \Rightarrow (\Gamma_2(K)) \Rightarrow (G_1)$

- [von Renesse & Sturm '05]
  X: 完備 Riemann 多様体, P_t: 熱半群なら OK:
  $(G_\infty) \Rightarrow \mathrm{Ric} \geq K \Rightarrow (W_\infty) \Rightarrow (G_1)$

- [Wang '05]
  一般には, $(G_2) \not\Rightarrow (G_1)$
  ($\mathbb{R}^m$ 上の SDE の解で反例が構成可？)

(3) 異質な評価

X: Lie 群

$\{X_i\}_{i=1}^n$: 左不変線型独立ベクトル場,

Hörmander 条件成立

- $P_t := \mathrm{e}^{t\mathcal{L}}, \quad \mathcal{L} := \sum_{i=1}^n X_i^2$
- $\Gamma(f,f) = \sum_{i=1}^n |X_i f|^2$, $|\nabla f| = \sqrt{\Gamma(f,f)}$

Bakry-Émery 型 (L^p-) 微分評価

$$|\nabla P_t f|(x) \leq C_p(t) P_t(|\nabla f|^p)(x)^{1/p} \qquad (\tilde{G}_p)$$

既知の結果

- 3-dim. Heisenberg 群, $C_p(t) \equiv C_p > 1$
  - $p > 1$: [Driver & Melcher '05]
  - $p = 1$: [H.-Q. Li '06] /
    [Bakry, Baudoin, Bonnefont & Chafaï '08]
- X: 一般, $p > 1$: [Melcher '08]
  (X: 冪零 $\Rightarrow C_q(t) \equiv C_p$)
- X: group of type H, $p = 1$, $C_p(t) \equiv C_p$:
  [Eldredge '10]
- $X = SU(2)$, $p > 1$, $C_p(t) = C_p \mathrm{e}^{-t}$:
  [Baudoin & Bonnefont '09]

注

★ $\mathfrak{L}$ に付随する距離 (Carnot-Carathéodory 距離) について,

$$(\tilde{G}_p)$$
$$\Updownarrow$$
$$W_{d,q}(P_t^*\mu, P_t^*\nu) \le C_p(t) W_{d,q}(\mu,\nu)$$

[K.'10]

!! 定数が $C_p(t) = C_p \mathrm{e}^{-Kt}$ with $C_p > 1$

$\Rightarrow$ 「$(\tilde{G}_p) \underset{\text{微分}}{\Rightarrow}$ "Γ_2-条件"」にはならない

Q.

$\tilde{G}_p$ の (確認が容易な) 十分条件は？

- 結合法は困難 (Heisenberg 群の場合でも未知)
  - [Kendall '09]
    Heisenberg 群上，違う種類のカップリング
- 結合法による (W_q) の証明自体，興味の対象
- [Baudoin, Garofalo, Bonnefont]
  Γ_2-条件の一般化
  ただし，($\tilde{G}_p$) は導出できていない