# Stein's method for invariant measures of diffusions via Malliavin calculus

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Nourdin and Peccati showed that by applying Stein's equation and Malliavin calculus we can measure the distance between the standard normal law and the distributions of random variables in $\mathbb{D}^{1,2}$ (the Sobolev space with respect to Malliavin derivative). This method is called Stein's method. And now, many mathematicians consider Stein's method with respect to other distributions instead of the standard normal law. For example, the Gamma distribution and the Pearson distribution. In this work, we consider the generalization of the method to more general distributions by using one-dimensional stochastic differential equations.

Let $S$ be the interval $(l,u)$ $(-\infty \leq l < u \leq \infty)$ and $\mu$ be a probability measure on $S$ with a density function $p$ which is continuous and strictly positive on $S$. Consider a continuous function $b$ on $S$ such that there exists $k \in (l,u)$ such that $b(x) > 0$ for $x \in (l,k)$, $b(x) < 0$ for $x \in (k,u)$, $bp$ is bounded on $S$ and

$$\int_l^u b(x)p(x)dx = 0.$$

Define

$$a(x) := \frac{2\int_l^x b(y)p(y)dy}{p(x)}, \quad x \in S.$$

Then, the stochastic differential equation:

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t, \quad t \geq 0$$

has a unique Markovian weak solution which is ergodic with invariant density $p$ (see [1]).

For $f \in C_0(S)$ (the set of continuous functions on $S$ vanishing at the boundary of $S$), let $m_f := \int_l^u f(x)p(x)dx$ and define $\tilde{g}_f$ by, for every $x \in S$,

$$\tilde{g}_f(x) := \frac{2}{a(x)p(x)} \int_l^x (f(y) - m_f)p(y)dy = -\frac{2}{a(x)p(x)} \int_x^u (f(y) - m_f)p(y)dy.$$

Then, $g_f(x) := \int_0^x \tilde{g}_f(y)dy$ satisfies that $f - m_f = Ag_f$ and by the definition of $m_f$ we have

$$f(x) - \mathbf{E}[f(X)] = \frac{1}{2}a(x)\tilde{g}_f'(x) + b(x)\tilde{g}_f(x)$$

where $X$ is a random variable with its law $\mu$. This equation is called Stein's equation. When $\mu$ has the standard normal distribution, $S$, $a$ and $b$ can be chosen as $(-\infty,\infty)$, 2 and $-x$, respectively.

We can estimate the bounds of $g_f$ and $g_f'$ as follows.

**Proposition 1.** *Assume that there exist $l', u' \in (l,u)$ such that $b$ is non-increasing on $(l,l')$ and $(u',u)$. Consider $f: S \to \mathbb{R}$ such that $\tilde{g}_f$ is well-defined and $\|f\|_\infty := \sup_{x\in S}|f(x)| < \infty$. Then we have*

$$\begin{aligned} ||\tilde{g}_f||_\infty &\leq C_1||f||_\infty \\ ||a\tilde{g}_f'||_\infty &\leq C_2||f||_\infty \end{aligned}$$

*where $C_1$ and $C_2$ are strictly positive constants.*

**Proposition 2.** *Assume that if $u < \infty$, there exists $u' \in (l,u)$ such that $b$ is non-decreasing and Lipschitz continuous on $[u',u)$ and $\liminf_{x\to u} a(x)/(u-x) > 0$; if $u = \infty$, there exists $u' \in (l,u)$ such that $b$ is non-decreasing on $[u',u)$ and $\liminf_{x\to u} a(x) > 0$. Similarly, assume that if $l < \infty$, there exists $l' \in (l,u)$ such that $b$ is non-increasing and Lipschitz continuous on $(l,l']$ and*

$\liminf_{x\to l} a(x)/(x-l) > 0$; *if* $l=-\infty$, *there exists* $l' \in (l,u)$ *such that* $b$ *is non-decreasing on* $(l,l']$ *and* $\liminf_{x\to l} a(x) > 0$. *Then we have*

$$||\tilde{g}_f'||_\infty \leq C_3(||f||_\infty + ||f'||_\infty)$$

*for* $f \in C_0^1(S)$ *where* $C_3$ *is a constant.*

We are now able to derive Stein's bound between the probability measure $\mu$ and the law of a random variable $Y$ in a certain class. Let $D$ be the Malliavin derivative, $D^*$ be the adjoint operator (the Skorohod integral operator) and $L := -D^*D$ (the Ornstein-Uhlenbeck operator).

**Theorem 3.** *Assume* $X \sim \mu$ *and let* $Y$ *be an* $S$*-valued random variable in* $\mathbb{D}^{1,2}$ *with* $b(Y) \in \mathbb{L}^2$. *Then for every* $f : S \to \mathbb{R}$ *such that* $\tilde{g}_f, \tilde{g}_f'$ *are bounded,*

$$\begin{aligned} &|\mathbf{E}[f(Y) - f(X)]| \\ &\leq ||\tilde{g}_f'||_\infty \mathbf{E}\left[\left|\mathbf{E}\left[\frac{1}{2}a(Y) + \langle D(-L)^{-1}\{b(Y) - \mathbf{E}[b(Y)]\}, DY\rangle_H \middle| Y\right]\right|\right] + ||\tilde{g}_f||_\infty |\mathbf{E}[b(Y)]|. \end{aligned}$$

The bound in Theorem 3 is optimal in the following sense.

**Theorem 4.** *A random variable* $Y \in \mathbb{D}^{1,2}$ *with its values on* $S$ *which satisfies* $b(Y) \in \mathbb{L}^2$. *Then,* $Y$ *has probability distribution* $\mu$ *if and only if* $\mathbf{E}[b(Y)] = 0$ *and*

$$\mathbf{E}\left[\frac{1}{2}a(Y) + \langle D(-L)^{-1}b(Y), DY\rangle_H \middle| Y\right] = 0.$$

Consider a distance between distributions of random variables $F$ and $G$ on $S$ defined by

$$d_{\mathcal{H}}(\mathcal{L}(F), \mathcal{L}(G)) := \sup_{f\in\mathcal{H}} |\mathbf{E}[f(F)] - \mathbf{E}[f(G)]| \tag{1}$$

where $\mathcal{L}(F)$ is the distribution of $F$ and $\mathcal{H}$ is a set of functions on $S$. By Theorem 3 we obtain an estimate for the distance between $X$ and $Y$ as follows:

$$\begin{aligned} d_{\mathcal{H}}(\mathcal{L}(Y), \mu) \leq \sup_{f\in\mathcal{F}} ||\tilde{g}_f'||_\infty \mathbf{E}\left[\left|\frac{1}{2}a(Y) + \langle D(-L)^{-1}\{b(Y) - \mathbf{E}[b(Y)]\}, DY\rangle_H\right|\right] \\ + \sup_{f\in\mathcal{F}} ||\tilde{g}_f||_\infty |\mathbf{E}[b(Y)]|. \end{aligned} \tag{2}$$

There are many kinds of distances between distributions defined by (1). For example, by taking $\mathcal{H} = \{1_{(l,z]}; z \in (l,u)\}$, one obtains the Kolmogorov distance; by taking $\mathcal{H} = \{f : ||f||_L \leq 1\}$, where $||\cdot||_L$ denotes the usual Lipschitz seminorm, one obtains the Wasserstein (or Kantorovich-Wasserstein) distance; by taking $\mathcal{H} = \{f : ||f||_{BL} \leq 1\}$, where $||\cdot||_{BL} = ||\cdot||_L + ||\cdot||_\infty$, one obtains the Fortet-Mourier (or bounded Wasserstein) distance; by taking $\mathcal{H}$ equal to the collection of all indicators $1_B$ of Borel sets, one obtains the total variation distance.

By (2), Propositions 1 and 2, we have estimates for the distances above. When $\inf_{x\in S} a(x) > 0$, by using Proposition 1, we have estimates for all the distances in the examples above. When $\inf_{x\in S} a(x) = 0$, by using Proposition 2, we have estimates for the Wasserstein distance and the Fortet-Mourier distance. Note that estimates for the Kolmogorov distance and the total variation distance are failed when $\inf_{x\in S} a(x) = 0$. If $Y$ is expressed as an explicit function of some Gaussian random variables, the bound in Theorem 3 can be calculated. Hence, we can apply these results to obtain the order of convergence for sequence of functions of Gaussian random variables.

# References


[1] B.M. Bibby, I.M. Skovgaard and M. Sørensen, Diffusion-type models with given marginals and auto-correlation function, *Bernoulli*, **11**(2), 2003 , 191-220.

[2] Seiichiro Kusuoka and Ciprian A. Tudor, Stein method for invariant measures of diffusions via Malliavin calculus, arXiv:1109.0684v1.