Strong uniqueness of diffusions to Gibbs measures on a path space with exponential interactions *

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In recent years, there has been a growing interest in the study of infinite dimensional stochastic dynamics associated with models of Euclidean quantum field theory, hydrodynamics, and statistical mechanics. Equilibrium states of such dynamics are described by Gibbs measures. The stochastic dynamics corresponding to these states is given by a diffusion semigroup. On some minimal domain of smooth functions, the infinitesimal generator of the semigroup coincides with the Dirichlet operator defined through a classical Dirichlet form of gradient type with a Gibbs measure. From an analytic point of view, it is very important to study L^p-uniqueness of the Dirichlet operator, that is, the question whether or not the Dirichlet operator restricted to the minimal domain has a unique closed extension in the L^p-space of the Gibbs measure under consideration, which generates a C_0-semigroup. As is well known, in the case of $p = 2$, this uniqueness is equivalent to essential self-adjointness.

The first objective of this talk is to prove L^p-uniqueness of the Dirichlet operator for all $p \geq 1$, under much weaker conditions on the growth rate of the potential function of the Gibbs measure by a modification of the SPDE approach presented in [5]. Important new examples are $\exp(\phi)_1$-quantum fields in infinite volume in the context of Euclidean quantum field theory. These models were introduced (for the case where $\mathbb{R}$ occurring in (1) below is replaced by a 2-dimensional Euclidean space-time $\mathbb{R}^2$ and where $d = 1$) by Albeverio and Høegh-Krohn in 1970's. More precisely, we are concerned with Gibbs measures on an infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$ given by the following formal expression:

$$\mu(dw) = Z^{-1} \exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} \big((-\Delta_x + m^2)w(x), w(x)\big)_{\mathbb{R}^d} dx \\ - \int_{\mathbb{R}} dx \Big(\int_{\mathbb{R}^d} e^{(w(x),\xi)_{\mathbb{R}^d}} \nu(d\xi)\Big)\Big\} \prod_{x\in\mathbb{R}} dw(x). \tag{1}$$

Here Z is a normalizing constant, $m > 0$ denotes mass, $\Delta_x := d^2/dx^2$, ν is a bounded positive measure on $\mathbb{R}^d$ with compact support, and $\prod_{x\in\mathbb{R}} dw(x)$ stands for a (heuristic) volume measure on the space of maps from $\mathbb{R}$ into $\mathbb{R}^d$. This has the interpretation of a quantized d-dimensional vector field with an interaction of exponential type in the 1-dimensional space-time $\mathbb{R}$, a model which is known as stochastic quantization of the $\exp(\phi)_1$-quantum field model (with weight measure ν).

The second objective of this talk is to discuss a characterization of the stochastic dynamics corresponding to the above Dirichlet operator. Due to general theory, the stochastic dynamics

*This talk is based on jointwork [2, 5] with Sergio Albeverio (Universität Bonn) and Michael Röckner (Universität Bielefeld).

constructed through the Dirichlet form approach solves the parabolic SPDE (5) weakly. However, we prove something much better, namely existence and uniqueness of a strong solution. We achieve this by first proving pathwise uniqueness for SPDE (5) and then applying the recent work of Ondreját [6] on the Yamada–Watanabe theorem for mild solutions of SPDE. As a consequence, we have the existence of a unique strong solution to SPDE (5) by using simple and straightforward arguments which do not rely on any finite volume approximations discussed as in Iwata [4] in case of polynomial (i.e., smooth) self-interaction.

Before stating our results, we introduce some notation and objects we will be working with. We define a weight function $\rho_r \in C^\infty(\mathbb{R},\mathbb{R}), r \in \mathbb{R}$, by $\rho_r(x) := e^{r\chi(x)}$, $x \in \mathbb{R}$, where $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. We fix a positive constant r sufficiently small. We set $E := L^2(\mathbb{R},\mathbb{R}^d; \rho_{-2r}(x)dx)$. This space is a Hilbert space with its inner product defined by

$$(w,\tilde{w})_E := \int_{\mathbb{R}} \big(w(x),\tilde{w}(x)\big)_{\mathbb{R}^d}\rho_{-2r}(x)dx, \quad w,\tilde{w} \in E.$$

Moreover, we set $H := L^2(\mathbb{R},\mathbb{R}^d)$ and denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ the corresponding norms in E and H, respectively. We regard the dual space E^* of E as $L^2(\mathbb{R},\mathbb{R}^d; \rho_{2r}(x)dx)$. We endow $C(\mathbb{R},\mathbb{R}^d)$ with the compact uniform topology and introduce a tempered subspace

$$\mathcal{C} := \{w \in C(\mathbb{R},\mathbb{R}^d) | \lim_{|x|\to\infty} |w(x)|\rho_{-r}(x) < \infty \text{ for every } r > 0\}.$$

We easily see that the inclusion $\mathcal{C} \subset E \cap C(\mathbb{R},\mathbb{R}^d)$ is dense with respect to the topology of E. Let $\mathcal{B}$ be the topological σ-field on $C(\mathbb{R},\mathbb{R}^d)$. For $T_1 < T_2 \in \mathbb{R}$, we define by $\mathcal{B}_{[T_1,T_2]}$ and $\mathcal{B}_{[T_1,T_2],c}$ the sub-σ-fields of $\mathcal{B}$ generated by $\{w(x); T_1 \leq x \leq T_2\}$ and $\{w(x); x \leq T_1, x \geq T_2\}$, respectively. For $T_1, T_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, let $\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}$ be the path space measure of the Brownian bridge such that $w(T_1) = z_1, w(T_2) = z_2$.

We now introduce a Gibbs measure μ on $C(\mathbb{R},\mathbb{R}^d)$. In this talk, we impose the following conditions on the potential function $U \in C(\mathbb{R}^d,\mathbb{R})$:

(U1) There exist a constant $K_1 \in \mathbb{R}$ and a convex function $V : \mathbb{R}^d \to \mathbb{R}$ such that

$$U(z) = \frac{K_1}{2}|z|^2 + V(z), \qquad z \in \mathbb{R}^d.$$

(U2) There exist $K_2 > 0$, $R > 0$ and $\alpha > 0$ such that

$$U(z) \geq K_2|z|^\alpha, \qquad |z| > R.$$

(U3) There exist $K_3, K_4 > 0$ and $0 < \beta < 1 + \frac{\alpha}{2}$ such that

$$|\widetilde{\nabla} U(z)| \leq K_3 \exp(K_4|z|^\beta), \qquad z \in \mathbb{R}^d,$$

where $\widetilde{\nabla} U(z) := K_1 z + \partial_0 V(z), z \in \mathbb{R}^d$ and $\partial_0 V$ is the minimal section of the subdifferential ∂V. (In the case where $U \in C^1(\mathbb{R}^d,\mathbb{R})$, $\widetilde{\nabla} U$ coincides with the usual gradient ∇U.)

Let $H_U := -\frac{1}{2}\Delta_z + U$ be the Schrödinger operator on $L^2(\mathbb{R}^d,\mathbb{R})$, where $\Delta_z := \sum_{i=1}^d \partial^2/\partial z_i^2$ is the d-dimensional Laplacian. Then condition **(U2)** assures that H_U has purely discrete spectrum and a complete set of eigenfunctions. We denote by $\lambda_0(>\min U)$ the minimal eigenvalue and by Ω the corresponding normalized eigenfunction in $L^2(\mathbb{R}^d,\mathbb{R})$. This eigenfunction is called ground state and it can be chosen to be strictly positive. Moreover, it has exponential decay at infinity. To be precise, there exist some positive constants D_1, D_2 such that

$$0<\Omega(z)\le D_1\exp\big(-D_2|z|U_{\frac{1}{2}|z|}(z)^{1/2}\big),\quad z\in\mathbb{R}^d,$$

where $U_{\frac{1}{2}|z|}(z):=\inf\{U(y)\mid |y-z|\le\frac{1}{2}|z|\}$.

For $T_1<T_2$, and for all $T_1\le x_1<x_2<\cdots<x_n\le T_2,\ A_1,A_2,\cdots,A_n\in\mathcal{B}(\mathbb{R}^d)$, we define a cylinder set $A\in\mathcal{B}_{[T_1,T_2]}$ by $A:=\{w\in C(\mathbb{R},\mathbb{R}^d)\mid w(x_1)\in A_1, w(x_2)\in A_2,\cdots,w(x_n)\in A_n\}$. Next, we set

$$\begin{aligned}\mu(A) &:= \Big(\Omega, e^{-(x_1-T_1)(H_U-\lambda_0)}\big(\mathbf{1}_{A_1}e^{-(x_2-x_1)(H_U-\lambda_0)}\big(\mathbf{1}_{A_2}\cdots\\ &\qquad\qquad e^{-(x_n-x_{n-1})(H_U-\lambda_0)}\big(\mathbf{1}_{A_n}e^{-(T_2-x_n)(H_U-\lambda_0)}\Omega\big)\big)\big)\Big)_{L^2(\mathbb{R}^d,\mathbb{R})}\\ &= e^{\lambda_0(T_2-T_1)}\int_{\mathbb{R}^d}dz_1\Omega(z_1)\int_{\mathbb{R}^d}dz_2\Omega(z_2)p(T_2-T_1,z_1,z_2)\\ &\quad\times\int_{C(\mathbb{R},\mathbb{R}^d)}\mathbf{1}_A(w)\exp\big(-\int_{T_1}^{T_2}U(w(x))dx\big)\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}(dw), \end{aligned}\tag{2}$$

where $p(t,z_1,z_2), t>0, z_1,z_2\in\mathbb{R}^d$, is the transition probability density of standard Brownian motion on $\mathbb{R}^d$, and we used the Feynman–Kac formula for the second line. Then by recalling that $e^{-tH_U}\Omega=e^{-t\lambda_0}\Omega$, $\|\Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})}=1$ and by the Markov property of the d-dimensional Brownian motion, (2) defines a consistent family of probability measures, and hence μ can be extended to a probability measure on $C(\mathbb{R},\mathbb{R}^d)$.

Furthermore, this measure satisfies the following DLR-equations:

$$\begin{aligned}\mathbb{E}^{\mu}\big[\mathbf{1}_A|\mathcal{B}_{[T_1,T_2],c}\big](\xi) &= Z^{-1}_{[T_1,T_2]}(\xi)\int_A\exp\Big(-\int_{T_1}^{T_2}U(w(x))dx\Big)\mathcal{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}(dw),\\ &\qquad \mu\text{-a.e. } \xi\in C(\mathbb{R},\mathbb{R}^d),\ \text{for all } A\in\mathcal{B}_{[T_1,T_2]}, T_1<T_2, \end{aligned}\tag{3}$$

where $Z_{[T_1,T_2]}(\xi):=\mathbb{E}^{\mathcal{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}}[\exp(-\int_{T_1}^{T_2}U(w(x))dx)]$ is a normalizing constant. (Although generally there exist other probability measures on $C(\mathbb{R},\mathbb{R}^d)$ satisfying the DLR-equation (3), we only consider the Gibbs measure μ which has been constructed in (2) in this talk.) We also note that the Gibbs measure μ is supported on $\mathcal{C}$ by using the standard moment estimates of Brownian motion. Then by the continuity of the inclusion map of $\mathcal{C}$ into E, we can regard $\mu\in\mathcal{P}(E)$ by identifying it with its image measure under the inclusion map.

Now we are in a position to introduce the pre-Dirichlet form $(\mathcal{E},\mathcal{FC}_b^\infty)$. Let $\mathcal{FC}_b^\infty$ be the space of all smooth cylinder functions on E having the form

$$F(w)=f(\langle w,\varphi_1\rangle,\dots,\langle w,\varphi_n\rangle),\quad w\in E,$$

with $n\in\mathbb{N}$, $f=f(\alpha_1,\dots,\alpha_n)\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ and $\varphi_1,\dots,\varphi_n\in C_0^\infty(\mathbb{R},\mathbb{R}^d)$. Here we set $\langle w,\varphi\rangle:=\int_{\mathbb{R}}(w(x),\varphi(x))_{\mathbb{R}^d}dx$ if the integral converges absolutely. Note that $\mathcal{FC}_b^\infty$ is dense in $L^p(\mu)$ for all

$p \geq 1$. For $F \in \mathcal{FC}_b^\infty$, we define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(w) := \sum_{j=1}^{n} \frac{\partial f}{\partial \alpha_j}(\langle w, \varphi_1\rangle, \ldots, \langle w, \varphi_n\rangle)\varphi_j.$$

Then we consider the pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ which is given by

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \big(D_H F(w), D_H G(w)\big)_H \mu(dw), \quad F, G \in \mathcal{FC}_b^\infty.$$

Proposition 1

$$\mathcal{E}(F,G) = -\int_E \mathcal{L}_0 F(w) G(w)\mu(dw), \quad F, G \in \mathcal{FC}_b^\infty,$$

where $\mathcal{L}_0 F \in L^p(\mu),\ p \geq 1,\ F \in \mathcal{FC}_b^\infty$, *is given by*

$$\begin{aligned}\mathcal{L}_0 F(w) \quad &= \quad \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle w, \varphi_1\rangle, \ldots, \langle w, \varphi_n\rangle\big)\langle \varphi_i, \varphi_j\rangle \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial f}{\partial \alpha_i}\big(\langle w, \varphi_1\rangle, \ldots, \langle w, \varphi_n\rangle\big) \cdot \big\{\langle w, \Delta_x \varphi_i\rangle - \langle (\widetilde{\nabla} U)(w(\cdot)), \varphi_i\rangle\big\}.\end{aligned}$$

This proposition means that the operator $\mathcal{L}_0$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable in $L^2(\mu)$. Let us denote by $\mathcal{D}(\mathcal{E})$ the completion of $\mathcal{FC}_b^\infty$ with respect to the $\mathcal{E}_1^{1/2}$-norm. By the standard theory of Dirichlet forms, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form and the operator $\mathcal{L}_0$ has a self-adjoint extension $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$, called the Friedrichs extension, corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. The semigroup $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ generated by $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$ is Markovian, i.e., $0 \leq e^{t\mathcal{L}_\mu} F \leq 1$, μ-a.e. whenever $0 \leq F \leq 1$, μ-a.e. Moreover, since $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ is symmetric on $L^2(\mu)$, the Markovian property implies that $\|e^{t\mathcal{L}_\mu} F\|_{L^1(\mu)} \leq \|F\|_{L^1(\mu)}$ holds for $F \in L^2(\mu)$, and $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ can be extended as a family of C_0-semigroup of contractions in $L^p(\mu)$ for all $p \geq 1$.

Theorem 1 (1) *The pre-Dirichlet operator* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *is* $L^p(\mu)$*-unique for all* $p \geq 1$, *i.e., there exists exactly one* C_0*-semigroup in* $L^p(\mu)$ *such that its generator extends* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$.
(2) *There exists a diffusion process* $\mathbb{M} := (\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbb{P}_w\}_{w\in E})$ *such that the semigroup* $\{P_t\}_{t\geq 0}$ *generated by the unique (self-adjoint) extension of* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *satisfies the following identity for any bounded measurable function* $F : E \to \mathbb{R}$, *and* $t > 0$*:*

$$P_t F(w) = \int_\Theta F(X_t(\omega))\mathbb{P}_w(d\omega), \quad \mu\text{-a.s. } w \in E.$$

Moreover, $\mathbb{M}$ *is the unique diffusion process solving the following "componentwise" SDE:*

$$\begin{aligned}\langle X_t, \varphi\rangle \quad &= \quad \langle w, \varphi\rangle + \langle B_t, \varphi\rangle + \frac{1}{2}\int_0^t \big\{\langle X_s, \Delta_x \varphi\rangle - \langle (\widetilde{\nabla} U)(X_s(\cdot)), \varphi\rangle\big\} ds, \\ &\qquad t > 0,\ \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d),\ \mathbb{P}_w\text{-a.s.,} \qquad (4)\end{aligned}$$

for quasi-every $w \in E$ *and such that its corresponding semigroup given by* (4) *consists of locally uniformly bounded (in* t*) operators on* $L^p(\mu), p \geq 1$, *where* $\{B_t\}_{t\geq 0}$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted* H*-cylindrical Brownian motion starting at zero defined on* $(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}_w)$.

Theorem 2 *For quasi-every $w \in E$, the parabolic SPDE*

$$dX_t(x) = \frac{1}{2}\{\Delta_x X_t(x) - (\widetilde{\nabla} U)(X_t(x))\}dt + dB_t(x), \quad x \in \mathbb{R},\ t > 0, \tag{5}$$

has a unique strong solution $X = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), E) \cap C((0,\infty), \mathcal{C})$. Namely, there exists a set $S \subset E$ with $\mathrm{Cap}(S) = 0$ such that for any H-cylindrical Brownian motion $\{B_t\}_{t\geq 0}$ starting at zero defined on a filtered probability space $(\Theta, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and an initial datum $w \in E \setminus S$, there exists a unique $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $X = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), E) \cap C((0,\infty), \mathcal{C})$ satisfying (4).

Remark 1 *Obviously, the uniqueness result in Theorem* 2 *implies the (thus weaker) uniqueness stated for the diffusion process $\mathbb{M}$ in Theorem* 1*. However, it does not imply the $L^p(\mu)$-uniqueness of the Dirichlet operator. This is obvious, since a priori the latter might have extensions which generate non-Markovian semigroups which thus have no probabilistic interpretation as transition probabilities of a process. Therefore, neither of the two uniqueness results in Theorems* 1 *and* 2*, i.e., $L^p(\mu)$-uniqueness of the Dirichlet operator and strong uniqueness of the corresponding SPDE respectively, implies the other.*

Remark 2 *If the potential function U is a C^1-function with polynomial growth at infinity, Iwata* [4] *proves that SPDE* (5) *has a unique strong solution $X^w = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), \mathcal{C})$ for every initial datum $w \in \mathcal{C}$. On the other hand, in the case of* $\exp(\phi)_1$*-quantum fields introduced below, since $(\nabla U)(w(\cdot)) \notin \mathcal{C}$ for $w \in \mathcal{C}$ in general, we cannot expect to solve SPDE* (5) *in $C([0,\infty), \mathcal{C})$ for a given initial datum $w \in \mathcal{C}$. Hence if we replace the state space $\mathcal{C}$ by a much smaller tempered subspace $\mathcal{C}_e$ such that $(\nabla U)(w(\cdot)) \in \mathcal{C}_e$ holds for $w \in \mathcal{C}_e$, we might construct a unique strong solution to SPDE* (5) *living in $C([0,\infty), \mathcal{C}_e)$ for every initial datum $w \in \mathcal{C}_e$. (A possible candidate for $\mathcal{C}_e$ could be the space of all paths behaving like*

$$|w(x)| \sim \log(\log(\log(\log(\cdots x)))), \quad |x| \to \infty.\ )$$

Example 1 ($P(\phi)_1$-quantum fields): We consider the case where the potential function U is written as $U(z) = \sum_{j=0}^{2n} a_j |z|^j, a_{2n} > 0, n \in \mathbb{N}$. A double-well potential $U(z) = a(|z|^4 - |z|^2), a > 0$, is also particularly important from the point of view of physics.

Example 2 ($\exp(\phi)_1$-quantum fields): We introduce a Gibbs measure μ with the formal expression (1). Let consider an exponential type potential function $U : \mathbb{R}^d \to \mathbb{R}$ (with weight ν) given by

$$U(z) = \frac{m^2}{2}|z|^2 + V(z) := \frac{m^2}{2}|z|^2 + \int_{\mathbb{R}^d} e^{(\xi,z)_{\mathbb{R}^d}} \nu(d\xi), \quad z \in \mathbb{R}^d,$$

where ν is a bounded positive measure with $\mathrm{supp}(\nu) \subset \{\xi \in \mathbb{R}^d |\ |\xi| \leq L\}$ for some $L > 0$. We note that U is a smooth strictly convex function (i.e., $\nabla^2 U \geq m^2$). Hence we can take $K_1 = m^2$, $K_2 = \frac{m^2}{2}$ and $\alpha = 2$. Moreover,

$$|U(z)| \leq \frac{m^2}{2}|z|^2 + \nu(\mathbb{R}^d)e^{L|z|} \leq \big(\frac{m^2}{2L^2} + \nu(\mathbb{R}^d)\big)e^{2L|z|}, \quad z \in \mathbb{R}^d,$$

and

$$|\nabla U(z)| \le m^2|z| + \int_{\mathbb{R}^d} |\xi| e^{(\xi,z)_{\mathbb{R}^d}} \nu(d\xi) \le (\frac{m^2}{L} + L\nu(\mathbb{R}^d))e^{L|z|}, \quad z \in \mathbb{R}^d.$$

Thus we can take $\beta = 1$, which satisfies $\beta < 1 + \frac{\alpha}{2}$ in condition **(U3)**.

Remark 3 *We discuss a simple example of* $\exp(\phi)_1$*-quantum fields in the case* $d = 1$*. This example has been discussed in the* 2*-dimensional space-time case in Albeverio–Høegh-Krohn* [1]*. Let* δ_a *be the Dirac measure at* $a \in \mathbb{R}$ *and we consider* $\nu(d\xi) := \frac{1}{2}\big(\delta_{-a}(d\xi) + \delta_a(d\xi)\big)$, $a > 0$*. Then the corresponding potential function is* $U(z) = \frac{m^2}{2}z^2 + \cosh(az)$*, and* (2) *implies that the Schrödinger operator* H_U *has a ground state* Ω *satisfying*

$$0 < \Omega(z) \le D_1 \exp\big(-\frac{D_2}{\sqrt{2}}|z|\, e^{\frac{a}{4}|z|}\big), \quad z \in \mathbb{R}, \tag{6}$$

for some $D_1, D_2 > 0$*. By the translation invariance of the Gibbs measure* μ *and* (6)*, there exist positive constants* M_1 *and* M_2 *such that*

$$\begin{aligned} A_T &:= \mu\big(\{w \in C(\mathbb{R},\mathbb{R}) |\ |w(T)| > \frac{4}{a}\log\log T\}\big) \\ &= \int_{|z|>\frac{4}{a}\log\log T} \Omega(z)^2 dz \le M_1 T^{-M_2 \log\log T} \end{aligned}$$

for T *large enough, and it implies* $\sum_{T=1}^{\infty} A_T < \infty$*. Then the first Borel–Cantelli lemma yields*

$$\mu\big(\{w \in C(\mathbb{R},\mathbb{R})|\ \limsup_{T\to\infty} \frac{|w(T)|}{\log\log T} \le \frac{4}{a}\}\big) = 1,$$

and thus μ *is supported by a much smaller subset of* $C(\mathbb{R},\mathbb{R})$ *than* $\mathcal{C}$*.*

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