Measurable Riemannian structure and its heat kernel analysis on the Sierpinski gasket

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On the Sierpinski gasket K (Figure 1), Kigami [2] has introduced the notion of the *'measurable Riemannian structure'* which is obtained by regarding K as a *'Riemannian submanifold in* $\mathbb{R}^2$*'* through a harmonic embedding (an injective harmonic map) $\Phi : K \to \mathbb{R}^2$. Then the image $K_{\mathcal{H}} := \Phi(K)$, called the *harmonic Sierpinski gasket* (Figure 2), can be considered as the geometric realization of the Riemannian structure, and there we have the analogues of the basic objects in Riemannian geometry like the gradient vector field $\widetilde{\nabla} u$ of a function u, the Riemannian volume measure μ and the geodesic metric $d_{\mathcal{H}}$. Moreover, by [2, Theorem 6.3], the associated heat kernel $p_t^{\mathcal{H}}(x,y)$ is subject to the two-sided Gaussian bound

$$\frac{c_1}{\mu\bigl(B_{\sqrt{t}}(x,d_{\mathcal{H}})\bigr)} \exp\biggl(-\frac{d_{\mathcal{H}}(x,y)^2}{c_2 t}\biggr) \leq p_t^{\mathcal{H}}(x,y) \leq \frac{c_3}{\mu\bigl(B_{\sqrt{t}}(x,d_{\mathcal{H}})\bigr)} \exp\biggl(-\frac{d_{\mathcal{H}}(x,y)^2}{c_4 t}\biggr) \tag{1}$$

in spite of the fractal nature of the space, where $B_r(x,d_{\mathcal{H}}) := \{y \in K \mid d_{\mathcal{H}}(x,y) < r\}$.

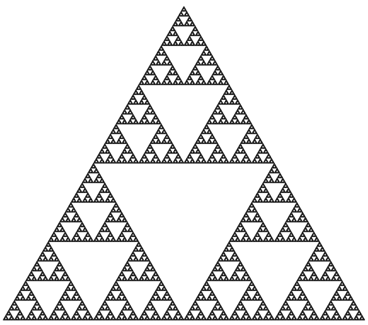

Figure 1: *The Sierpinski gasket*

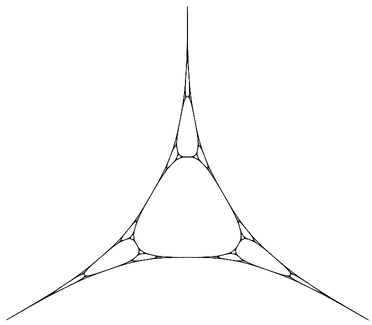

Figure 2: *The harmonic Sierpinski gasket*

In this talk I will present various detailed short time asymptotic behaviors of $p_t^{\mathcal{H}}(x,y)$, which have been established in a recent preprint [1]. The first main result is the following off-diagonal Gaussian behavior, so-called *Varadhan's asymptotic relation*:

Theorem 1 ([1, Corollary 4.4]) *For any* $x, y \in K$,

$$\lim_{t\downarrow 0} 2t \log p_t^{\mathcal{H}}(x,y) = -d_{\mathcal{H}}(x,y)^2. \tag{2}$$

Definition 2 For $x, y \in K$, define $d_\mu(x,y)$ by

$$d_\mu(x,y) := \sup\{u(x) - u(y) \mid u \in \mathcal{F},\ |\widetilde{\nabla} u| \leq 1\ \mu\text{-a.e.}\}, \tag{3}$$

where $\mathcal{F}$ denotes the *domain of the standard Dirichlet form on* K (note that $\mathcal{F} \subset C(K)$).

The results of Sturm [*J. Math. Pures Appl.* **75** (1996), 273–297] and Ramírez [*Comm. Pure Appl. Math.* **54** (2001), 259–293] tell us that the limit in the left-hand side of (2) exists and is equal to $d_\mu(x,y)$. Therefore the essence of Theorem 1 consists in the following identification of the distance functions $d_{\mathcal{H}}$ and d_μ.

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Theorem 3 ([1, Theorem 4.2])

$$d_{\mathcal{H}} = d_\mu. \tag{4}$$

For on-(and near-)diagonal behaviors of $p_t^{\mathcal{H}}(x,y)$, we first have the following theorem.

Theorem 4 ([1, Section 5]) *For $m \in \mathbb{N} \cup \{0\}$ let V_m denote the set of vertices of level m, so that their union $V_* := \bigcup_{m\in\mathbb{N}} V_m$ is dense in K. Then for any $x \in V_*$ we have the following:*
(1) There exists an explicit constant $\xi_x \in (0,\infty)$ determined solely by x, such that

$$p_t^{\mathcal{H}}(x,x) = \frac{1}{\xi_x\sqrt{2\pi t}}\bigl(1 + O(t^{\log_{5/3}3})\bigr) \qquad \text{and} \qquad \lim_{r\downarrow 0}\frac{\mu(B_r(x,d_{\mathcal{H}}))}{r} = 2\xi_x. \tag{5}$$

Furthermore there exist $c_x, t_x, r_x \in (0,\infty)$ such that for $\delta \in (0,1]$, $t \in (0,t_x]$ and $y \in B_{r_x}(x, d_{\mathcal{H}})$,

$$\left| p_t^{\mathcal{H}}(x,y) - \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} \right| \le \frac{c_x}{\delta^{12}}\left(t^{\log_{5/3}3} + |h_x(y)|^{\frac{4\log_{5/3}3}{2+\log_5 3}} \right)\frac{\exp\bigl(-\frac{h_x(y)^2}{2(1+\delta)t}\bigr)}{\sqrt{2\pi t}}, \tag{6}$$

where h_x denotes the coordinate along the 'tangent line of $K_{\mathcal{H}}$ at x' such that $h_x(x) = 0$.
(2) Let $X = (\{X_t\}_{t\ge 0}, \{\mathbf{P}_x\}_{x\in K})$ be the corresponding diffusion on K. Then for any $\alpha \in (-1,\infty)$,

$$\lim_{t\downarrow 0}\frac{\mathbf{E}_x[d_{\mathcal{H}}(x,X_t)^\alpha]}{t^{\alpha/2}} = \int_{\mathbb{R}} |y|^\alpha \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy. \tag{7}$$

Theorem 4 says that the heat kernel $p_t^{\mathcal{H}}(x,y)$, as well as the measures $\mu(B_r(x,d_{\mathcal{H}}))$ of geodesic balls and the moments $\mathbf{E}_x[d_{\mathcal{H}}(x,X_t)^\alpha]$ of displacement of the diffusion, exhibits **one-dimensional** behaviors at each vertex $x \in V_*$. (Note that V_* is countable and hence $\mu(V_*) = 0$.)

On the other hand, at *'generic'* points x, the short time behavior of $p_t^{\mathcal{H}}(x,x)$ is **non-integer-dimensional**, as follows.

Theorem 5 ([1, Theorem 6.1]) *There exists $d_{\mathrm{S}}^{\mathrm{loc}} \in \mathbb{R}$, $1 < d_{\mathrm{S}}^{\mathrm{loc}} \le 2\log_{25/3}5 = 1.5181\ldots$, such that*

$$\lim_{t\downarrow 0}\frac{2\log p_t^{\mathcal{H}}(x,x)}{-\log t} = d_{\mathrm{S}}^{\mathrm{loc}} \qquad \mu\text{-a.e. } x \in K. \tag{8}$$

Finally, the following theorem shows a non-integer-dimensional asymptotic behavior of the eigenvalues of the associated Laplacian.

Theorem 6 ([1, Theorem 7.2]) *For the eigenvalues $\{\lambda_n^{\mathcal{H}}\}_{n\in\mathbb{N}}$ of the associated Laplacian and $s,t > 0$, let $\mathcal{N}_{\mathcal{H}}(s) := \#\{n \in \mathbb{N} \mid \lambda_n^{\mathcal{H}} \le s\}$ and $\mathcal{Z}_{\mathcal{H}}(t) := \sum_{n\in\mathbb{N}} e^{-\lambda_n^{\mathcal{H}}t} = \int_K p_t^{\mathcal{H}}(x,x)d\mu(x)$. Then*

$$\lim_{s\to\infty}\frac{2\log \mathcal{N}_{\mathcal{H}}(s)}{\log s} = \lim_{t\downarrow 0}\frac{2\log \mathcal{Z}_{\mathcal{H}}(t)}{-\log t} = \dim_{\mathrm{H}}(K,d_{\mathcal{H}}) = \dim_{\mathrm{B}}(K,d_{\mathcal{H}}) \in [d_{\mathrm{S}}^{\mathrm{loc}}, 2\log_{25/3}5], \tag{9}$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{B}}$ denote respectively Hausdorff and box-counting dimensions.

References

[1] N. Kajino, *Heat kernel asymptotics for the measurable Riemannian structure on the Sierpinski gasket*, preprint, 2010.

[2] J. Kigami, Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate, *Math. Ann.* **340** (2008), 781–804.