Wiener-Poisson 空間上のマリアバン解析とその伊藤型SDEへの応用

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1 Introduction

1.1 Sobolev spaces on the Wiener-Poisson space

Probability space. We set $(\Omega, \mathcal{F})$ to be the product measure space (i.e., $\Omega = \Omega_1 \times \Omega_2, \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$), and denote $\omega = (\omega_1, \omega_2) \in \Omega$. We consider the product probability measure $\mathbf{P} = \mathbf{P_1} \otimes \mathbf{P_2}$ on $(\Omega, \mathcal{F})$. The completion of $\mathcal{F}$ with respect to $\mathbf{P}$ is denoted by $\bar{\mathcal{F}}$. Sub σ-fields $\mathcal{F}_1 \otimes \{\phi, \Omega_2\}$ and $\{\phi, \Omega_1\} \otimes \mathcal{F}_2$ are identified with $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. We set $W(t)(\omega) = W(t)(\omega_1) = \omega_1(t), N(dtdz)(\omega) = N(dtdz)(\omega_2) = \omega_2(dtdz)$. The triplet $(\Omega, \bar{\mathcal{F}}, \mathbf{P})$ is called the *Wiener-Poisson space with Lévy measure* μ.

The operator $D_{(t,u)}$. We define $\mathbf{K} := \mathbf{K}_1 \oplus \mathbf{K}_2$. Then $\mathbf{K}$ is a Hilbert space with the inner product

$$(h_1, h_2) = (f_1, f_2)_{\mathbf{K}_1} + (g_1, g_2)_{\mathbf{K}_2}$$

where $h_i = f_i \oplus g_i \in \mathbf{K}$. We regard the operators D_t and $\tilde{D}_u$ as $D_t \otimes id$ and $id \otimes \tilde{D}_u$, respectively. It should be noticed that the operators $\varepsilon_u^{\pm}$ are also extended to Ω by setting $\varepsilon_u^{\pm}(\omega_1, \omega_2) = (\omega_1, \varepsilon_u^{\pm}\omega_2)$. Let us introduce the operator $D_{(t,u)}$ as

$$D_{(t,u)} := D_t \oplus \tilde{D}_u \,:\, L^2(\Omega) \to L^2(\Omega\,;\,\mathbf{K});$$

for $X = \sum_{i=1}^k X_1^{(i)} X_2^{(i)} \in \mathcal{P}$ where $X_1^{(i)} \in L^2(\Omega_1)$ and $X_2^{(i)} \in L^2(\Omega_2)$, we have

$$D_{(t,u)}X = \sum_{i=k} \left(D_t X_1^{(i)} X_2^{(i)} \oplus X_1^{(i)} \tilde{D}_u X_2^{(i)} \right),$$

if $X_1^{(i)}$ and $X_2^{(i)}$ are in the domain of D and of $\tilde{D}$, respectively. The operator $D_{(t,u)}$ is a closed and unbounded operator.

For $\mathbf{t} = (t_1, ..., t_k), \mathbf{u} = (u_1, ..., u_k)$, let

$$D_{(\mathbf{t},\mathbf{u})} = D_{(t_k,u_k)} \cdots D_{(t_2,u_2)} D_{(t_1,u_1)}.$$

Let $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$. Spaces $\mathcal{P}_1, \mathcal{P}_2$ are identifies with $\mathcal{P}_1 \otimes 1, 1 \otimes \mathcal{P}_2$ respectively. We put for $p \geq 2$

$$\mathbf{D}_{k,l,p} = \bar{\mathcal{P}}^{|\cdot|_{k,l,p}},$$

where

$$|F|_{k,l,p} := \left(|F|_{0,l,p}^p + \sum_{k'=1}^k \sum_{l'=0}^l \mathbf{E}\left[\int_{\mathbf{A}(\rho)^{\mathbf{k'}}} \left(\int_{\mathbf{T}^{\mathbf{l'}}} \left| \frac{\mathbf{D_t^{l'} \tilde{D}_u^{k'} F}}{\gamma(\mathbf{u})} \right|^{\mathbf{2}} \mathbf{dt} \right)^{\mathbf{p/2}} \hat{\mathbf{M}}(\mathbf{du}) \right] \right)^{\mathbf{1/p}}. \quad (1.1)$$

[1] Some parts of this talk are based on joint works with M. Hayashi and with Prof. H. Kunita.

The operator $D_{(t,u)}$ is extended continuously to $\mathbf{D}_{k,l,p}$. Let

$$\mathbf{D}_\infty = \cap_{k,l=0}^{\infty} \cap_{p\geq 2} \mathbf{D}_{k,l,p}.$$

Among smooth functionals, we are particularly interested in *nondegerate functionals* in the sense of Malliavin-Picard (functionals satisfying the Condition **(ND)** stated below). An important property of a nondegenerate functional F is that for any $n \in \mathbf{N}$ there exist $k,l \in \mathbf{N}, p > 2$ and $C > 0$ such that for any $\xi \in \mathbf{R}^d$

$$|E[e^{i\xi\cdot F}G]| \leq C(1+|\xi|^2)^{-q_0 n/2}|G|_{k,l,p}, \tag{1.2}$$

where $0 < q_0 < 1$ is a constant independent of n. Putting $G = 1$, this inequality shows that the characteristic function of F satisfies

$$E[e^{i\xi\cdot F}] = O((1+|\xi|^2)^{-q_0 n/2}) \tag{1.3}$$

as $|\xi| \to \infty$ for any n. This implies that F has a C^∞-density function.

Let $\mathcal{S}$ be the set of all rapidly decreasing C^∞-functions and let $\mathcal{S}'$ be the set of tempered distributions. Another important consequence of (1.2) is that for all $s > 0$ there exist $k,l \in \mathbf{N}, p > 2$ and a positive constant c such that for any $\phi \in \mathcal{S}$ the inequality

$$|\phi(F)|'_{k,l,p} \leq c|\mathcal{F}\phi|_{\mathbf{H}^{-s}} \tag{1.4}$$

holds. Here $(\mathbf{H}^s, |\cdot|_{\mathbf{H}^s}), s \in \mathbf{R}$ are (weighted) Sobolev spaces on $\mathbf{R}^d$ such that $\cup_{s>0}\mathbf{H}^{-s} \subset \mathcal{S}'$, and $\mathcal{F}$ denotes the Fourier transform. This equality enables us to extend the composition $\phi \circ F$ to $T \circ F$ for a tempered distribution T, so that $T \circ F$ is defined as a generalized functional.

2 Itô type case

In this section we choose F to be X_t, where $(t \mapsto X_t)$ denotes a Itô type process and $t > 0$. We show the existence of the smooth density under such conditions..

A Itô type process X_t is given by the S.D.E.

$$X_t = \int_0^t b(X_r)dr + \int_0^t \sigma(X_r)dW_r + \int_0^t \int g(X_r, z)\tilde{N}(drdz),$$

$$X_0 = x.$$

Here the coefficients b, σ and g are infinitely times continuously differentialble with respect to x. All derivatives of b, σ of all orders are assumed to be bounded.

2.1 Decomposition

We assume the function g satisfies

$$|g(x,z)| \leq (1+|x|)K^0(z),$$

$$|\nabla_x g(x,z)|| \le K^1(z)$$

for some positive functions $K^0(z), K^1(z)$ such that $\int K^i(z)\mu(dz) < +\infty$ for $p \ge 2, i = 0, 1$.

The flow condition means

$$\inf_{x,z} |\det(1 + \nabla g(x,z))| > C > 0.$$

A sufficient condition for the flow property is
Condition (D):

$$\sup_{x,z} |\nabla g(x,z)| \le \frac{1}{2}. \tag{2.1}$$

We choose $\delta_0 > 0$ so that (2.1) holds for $|z| < \delta_0$.

We decompose the Lévy measure μ into the sum $\mu' + \mu''$, where

$$\mu'(dz) = 1_{(0,\delta_0]}(|z|)\mu(dz),$$

$$\mu''(dz) = 1_{(\delta_0,+\infty)}(|z|)\mu(dz).$$

Accordingly we decompose $N(dtdz)$ as $N'(dtdz) + N''(dtdz)$.

We decompose the probability space $(\Omega, \mathcal{F}, P)$ as follows: $\Omega = \Omega' \times \Omega'', \mathcal{F} = \mathcal{F}' \times \mathcal{F}'', P = P' \times P''$

Let z''_t be a Lévy process given by $N''(dtdz)$ defined on $(\Omega'', \mathcal{F}'', P'')$.

We denote by X'_t the solution to the S.D.E.

$$X'_t = \int_0^t (b(X'_r) - \int_{\delta_0 < |z| < 1} g(X'_r, z)\mu(dz))dr$$

$$+ \int_0^t \sigma(X'_r)dW_r + \int_0^t \int_{|z| \le \delta_0} g(X'_r, z)\tilde{N}'(drdz),$$

$$X'_0 = x.$$

We write $X'_{s,t} = X'_t \circ (X'_s)^{-1}$ for $s < t$. Then X_t is a solution to the S.D.E. driven by

$$X'_t \oplus \int_0^t \int_{|z| > \delta_0} g(x,r,z)N(drdz).$$

We assume, for any positive integer k and $p > 1$ derivatives satisfy

$$\sup_{t \in T, \mathbf{u} \in A(1)^k} E\left[\sum_{i=1}^m \sup_{|z_i| \le 1} |\partial_{z_i} \tilde{D}_{t,z} X' \circ \epsilon^+_{\mathbf{u}}|^p + \sum_{i,j=1}^m \sup_{|z| \le 1} |\partial_{z_i}\partial_{z_j} \tilde{D}_{t,z} X' \circ \epsilon^+_{\mathbf{u}}|^p\right] < \infty. \tag{2.2}$$

2.2 Use of Malliavin calculus for small jumps

Let $F' = X'_t(x)$. Then

$$F' \circ \epsilon^+_{t,z} = X'_{t,T} \circ \phi_{t,z} \circ X'_{t-}(x).$$

Hence

$$\begin{aligned}\tilde{D}_{t,z}F' &= X'_{t,T} \circ \phi_{t,z}(X'_{t-}) - X'_{t,T}(X'_{t-}) \\ &= \nabla X'_{t,T}(\phi_{t,\theta z}(X'_{t-}))\nabla_z \phi_{t,\theta z}(X'_{t-})\end{aligned}$$

for some $\theta \in (0,1)$.

This implies $F' \in \mathbf{D}_{1,0,p}$, and finally $F' \in \mathbf{D}_\infty$.

Further we have

$$\partial_z \tilde{D}_{t,z}F'|_{z=0} = \nabla X'_{t,T}(X'_{t-})G(X'_{t-},t).$$

In view of this we put for $0 < \rho < 1$

$$\tilde{K}_\rho \equiv \int_0^T \nabla X'_{t,T}(X'_{t-})G(X'_{t-})B_\rho G(X'_{t-})^T \nabla X'_{t,T}(X'_{t-})^T dt,$$

and

$$R^x \equiv \int_0^T \nabla X'_{t,T}(X'_{t-})\sigma(X'_{t-})\sigma(X'_{t-})^T \nabla X'_{t,T}(X'_{t-})^T dt$$

Then

$$R^x + \tilde{K}_\rho = \int_0^T \nabla X'_{t,T}(X'_{t-})C_\rho(X'_{t-})\nabla X'_{t,T}(X'_{t-})^T dt,$$

where

$$C_\rho(x) = \sigma(x)\sigma(x)^T + G(x)B_\rho G(x)^T.$$

Definition 1 *(1) We say F satisfies the* **(NDB)** *condition if there exists $0 < \rho_0$ such that*

$$(v, C_\rho(x)v) \geq C|v|^2 \tag{2.3}$$

holds for all x, t and $0 < \rho < \rho_0$.

(2) We say that F satisfies the **(ND)** *condition if for all $p \geq 1, k \geq 0$ there exists $\beta \in (\frac{\alpha}{2}, 1]$ such that*

$$\sup_{\rho\in(0,1)} \sup_{\substack{v\in\mathbf{R}^d,\\ |v|=1}} \sup_{\tau\in A^k(\rho)} E[\left|\left((v,\Sigma v) + \varphi(\rho)^{-1}\int_{A(\rho)} |(v,\tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F|\leq\rho^\beta\}}\hat{N}(du)\right)^{-1} \circ \epsilon^+_\tau\right|^p] < \infty,$$

where Σ is the Malliavin's covariance matrix $\Sigma = (\Sigma_{i,j})$, where $\Sigma_{i,j} = \int_T (D_t F_i, D_t F_j)dt$.

Proposition 1 *Assume the conditions* **(NDB)** *and* **(D)**. *Then for each $N \in \mathbf{N}$ the family $\{X'_t(x)\}_{x\in\{|x|\leq N\}}$ satisfies the* **(ND)** *condition.*

This implies that the estimate (1.2) is justified for $F' = X'_{t_0}$. Since F' and N'' are independent, putting $e_v(x) = e^{i(x,v)}$, we have for any n there exist k, l, p and $C > 0$ such that

$$E[Ge_v(F)] = E''[E'[e_v(F' \circ \epsilon_q^+)]] \tag{2.4}$$
$$\leq C(1 + |v|^2)^{-\frac{1}{2}nq_0} E''[\{|F' \circ \epsilon_q^+|_{k,l,p}^n \times |\tilde{Q}_\rho'^{-1}(v) \circ \epsilon_q^+|_{k,l,p}^n\}],$$

where $\tilde{Q}'_\rho(v) = (v', (R + \tilde{K}_\rho)v')$.

Our objective below is to show the finiteness of the R.H.S. of (2.4) under the conditions **(NDB)** and **(D)**.

References

[1] M. Hayashi and Y. Ishikawa, Composition with distributions of Wiener-Poisson variables and its asymptotic expansion, Math. Nach. to appear.

[2] Ishikawa, Y. and Kunita, H., Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps, Stochastic processes and their applications 116 (2006) 1743–1769.