Hypoellipticity and ergodicity of the Wonham filter as a diffusion process

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1 Introduction

The ergodicity problem of the Wonham filter as a diffusion process is discussed. We show that an ergodic theorem of degenerate Markov diffusions is applicable to the problem even when the Wonham equation is degenerate. Under a certain condition, the Wonham equation satisfies Hörmander's condition and the Wonham filter has a continuous transition density. From these results, we obtain that the transition kernel of the Wonham filter as a diffusion process converges to a unique invariant probability measure as $t \to \infty$ under the condition.

2 Settings

Let us introduce a pair of continuous time stochastic processes (X_t, Y_t), where X_t and Y_t represent the **signal** and the **noisy observation** of X_t respectively. X is a finite-state continuous-time Markov chain with finite state space $\mathcal{E}^d = \{e_1, \cdots, e_d\}$ and a Q-matrix $\Lambda := (\lambda_{ij})_{1 \le i,j \le d}$ as a generator. Y is a **one-dimensional** process defined by

$$\text{observation} \qquad Y_t := \int_0^t g(X_s)ds + \sigma B_t, \tag{1}$$

where $g : \mathcal{E}^d \longrightarrow \mathbb{R}$, $\sigma > 0$. The observation noise B_t is a **one-dimensional** Brownian motion independent of X_t. Set $g_i := g(e_i)$. Let G be a diagonal matrix with $G_{ii} := g_i$. Without loss of generality, we can assume that $g_i \le g_j$ if $i \le j$, and that $0 \le g_i$ for each $1 \le i \le d$. We are interested in the asymptotic behaviour of the conditional distribution

$$p_t := (p_t^1, \cdots, p_t^d), \quad p_t^i := P(X_t = e_i | \mathcal{Y}_t), \tag{2}$$

where $\mathcal{Y}_t := \sigma(Y_s : 0 \le s \le t)$. The dynamics of p_t is described by the following stochastic differential equation (SDE),

$$\begin{aligned} dp_t^i &= (\Lambda^* p_t)^i dt + p_t^i \{g_i - p_t(g)\} \sigma^{-2} \{dY_t - p_t(g)dt\}, \\ p_0^i &= \beta^i, \quad i = 1, \cdots, d, \end{aligned} \tag{3}$$

where $p_t(g) = \sum_{j=1}^d g(e_j) p_t^j$. This SDE is called *the Wonham equation*. It is well-known that the process

$$W_t := \sigma^{-1}(Y_t - \int_0^t p_s(g)ds)$$

is a **one-dimensional** Brownian motion adapted to $\mathcal{Y}_t$. Therefore we can rewrite (3) as

$$\begin{aligned} dp_t^i &= (\Lambda^* p_t)^i dt + p_t^i \{g_i - p_t(g)\}\sigma^{-1} dW_t \\ &=: U_0^i(p_t)dt + V^i(p_t)dW_t \\ &= U^i(p_t)dt + V^i(p_t) \circ dW_t, \end{aligned} \tag{4}$$

where $U^i(p) := U_0^i(p) - 1/2 \sum_{j=1}^d V^j \partial_j V^i(p)$. We define

$$\begin{aligned} \mathcal{S}_d &:= \{x \in \mathbb{R}^d \mid 0 \le x_i \le 1, \textstyle\sum_{i=1}^d x_i = 1\} \text{ (simplex)}, \\ \mathcal{I}_d &:= \{x \in \mathbb{R}^d \mid 0 \le x_i \le 1, \textstyle\sum_{i=1}^d x_i \le 1\} \text{ (interior)}. \end{aligned}$$

Notice that $\sum_{i=1}^d p_t^i \equiv 1$ holds and p_t is a diffusion on $\mathcal{S}_d$. Hence we study the $d-1$-dimensional diffusion

$$q_t := (p_t^1, \dots, p_t^{d-1})$$

in $\mathcal{I}_{d-1}$, and will show its ergodicity and so on. If q_t is ergodic in $\mathcal{I}_{d-1}$, we say that *the Wonham filter is ergodic*. The filter q_t satisfies the following SDE;

$$dq_t^i = A^i(q_t)dt + B^i(q_t) \circ dW_t \tag{5}$$

for $1 \le i \le d-1$, where

$$\begin{aligned} A^i(q_1, \dots, q_{d-1}) &:= U^i(q_1, \dots, q_{d-1}, 1 - \textstyle\sum_{i=1}^{d-1} q_i), \\ B^i(q_1, \dots, q_{d-1}) &:= V^i(q_1, \dots, q_{d-1}, 1 - \textstyle\sum_{i=1}^{d-1} q_i). \end{aligned}$$

Let $Q_t(q, \cdot)$ be the transition kernel of q_t. We show that the vector fields, A^i, B^i, satisfy Hörmander's condition under a certain condition.

3 Main Result

Here we state our main result.

Assumption 1. *Q-matrix Λ is irreducible. $g_i - g_j \neq g_l - g_k$ holds if $(i,j) \neq (l,k)$.*

Assumption 2. *Q-matrix Λ is irreducible, $\lambda_{1d} \neq 0$, and $\lambda_{d1} \neq 0$. $g_i < g_j$ holds if $i < j$.*

Theorem 3 (Main Theorem)**.** *Assume Assumption 1 or Assumption 2. Then q_t has a continuous transition density on $\mathcal{I}_{d-1}$ and a unique invariant probability measure ν on $\mathcal{I}_{d-1}$. Moreover it is exponentially ergodic on $\mathcal{I}_{d-1}$,*

$$\|Q_t(q, \cdot) - \nu\| \le R_1 e^{-\alpha t} \quad \text{for all } q \in \mathcal{I}_{d-1},$$

i.e., the Wonham filter is exponentially ergodic.

Here $\| \cdot \|$ denotes the total variation of measures on $\mathcal{I}_{d-1}$.