Infinitely dimensional stochastic differential equations for interacting Brownian motions

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Let $S=\mathbb{R}^d$ and let S be the configuration space over S. Let $\sigma : S\times\mathsf{S}\to\mathbb{R}^{d^2}$ and $\mathbf{b} : S\times\mathsf{S}\to\mathbb{R}^d\cup\{\mathbf{\Delta}\}$ be measurable functions. Here $\mathbf{\Delta}$ means an extra point. Let $\mathbf{a}=\sigma\sigma^t$. We assume there exists a positive constant c_1 independent of (x,x) for each $(x,\mathsf{x})\in S\times\mathsf{S}$ such that

$$0\le\sum_{m,n=1}^{d}\mathbf{a}_{mn}(x,\mathsf{x})\xi_m\xi_n\le c_1|\xi|^2\quad\text{for all }\xi=(\xi_m)\in\mathbb{R}^d. \tag{1}$$

For $\mathbf{X}=(X^i)_{i\in\mathbb{N}}$ we set $(X^i,\mathsf{X}^{i*})=\{(X^i_t,\mathsf{X}^{i*}_t)\}\in C([0,\infty);S\times\mathsf{S})$ by

$$(X^i_t,\mathsf{X}^{i*}_t)=(X^i_t,\sum_{j\ne i,j\in\mathbb{N}}\delta_{X^j_t}).$$

We study the SDEs of the form:

$$dX^i_t=\sigma(X^i_t,\mathsf{X}^{i*}_t)dB^i_t+\mathbf{b}(X^i_t,\mathsf{X}^{i*}_t)dt\quad(i\in\mathbb{N}). \tag{2}$$

Let $S^\infty=S^{\mathbb{N}}$. Let $\check{\sigma}$ and $\check{\mathbf{b}}$ be the functions of $(x,(x_j)_{j\in\mathbb{N}})$ defined on $S\times S^\infty$ being symmetric in $(x_j)_{j\in\mathbb{N}}$ for each x and satisfying

$$\check{\sigma}(x,(x_j)_{j\in\mathbb{N}})=\sigma(x,\sum_{j\in\mathbb{N}}\delta_{x_j}),\quad\check{\mathbf{b}}(x,(x_j)_{j\in\mathbb{N}})=\mathbf{b}(x,\sum_{j\in\mathbb{N}}\delta_{x_j}). \tag{3}$$

Then we can rewrite (2) as (4)

$$dX^i_t=\check{\sigma}(X^i_t,(X^j_t)_{j\ne i})dB^i_t+\check{\mathbf{b}}(X^i_t,(X^j_t)_{j\ne i})dt\quad(i\in\mathbb{N}). \tag{4}$$

Let $\check{\mathbf{a}}=\check{\sigma}\check{\sigma}^t$. Write $\check{\mathbf{a}}=[\check{\mathbf{a}}_{kl}]_{1\le k,l\le d}$ and $\check{\mathbf{b}}=(\check{\mathbf{b}}_k)_{1\le k\le d}$. Then intuitively the generator is

$$\mathbf{L}:=\frac{1}{2}\sum_{i\in\mathbb{N}}\sum_{k,l=1}^{d}\check{\mathbf{a}}_{kl}(s_i,(s_j)_{j\in\mathbb{I},j\ne i})\frac{\partial^2}{\partial s_{ik}\partial s_{il}}+\sum_{i\in\mathbb{N}}\sum_{k=1}^{d}\check{\mathbf{b}}_k(s_i,(s_j)_{j\in\mathbb{I},j\ne i})\frac{\partial}{\partial s_{ik}}. \tag{5}$$

Here $s_i=(s_{i1},\dots,s_{id})\in\mathbb{R}^d$.

Our strategy to solve SDE (2) (and (4)) is to use a geometric property behind the SDE (2). We first consider invariant probability measure μ of the unlabeled dynamics associated with (2). Namely, we consider a probability measure μ whose log derivatve $\mathbf{d}^\mu$ satisfies $\mathbf{b}(x,\mathsf{y})=\{\nabla_x\mathbf{a}(x,\mathsf{y})+\mathbf{a}(x,\mathsf{y})\mathbf{d}^\mu(x,\mathsf{y})\}/2$. Here $\mathbf{d}^\mu$ is the log derivative of the measure μ^1 given by (6), and the definition of $\mathbf{d}^\mu$ is given by (12).

Note that for a given pair $(\mathbf{a},\mu)$, $\mathbf{b}$ is determined uniquely. We construct the unlabeled diffusion associated with $(\mathbf{a},\mu)$ by using the Dirichlet space given by $(\mathbf{a},\mu)$ and prove the labeled process consisting of each component of the unlabeled diffusion satisfies (2) and (4).

If there were a Dirichlet space associated with the the (fully) labeled diffusion $\mathbf{X}$, we could use Ito formula for each component X^i and X^iX^j, and prove $\mathbf{X}$ satisfies (5) since all coordinate functions x^i,x^ix^j $(i,j\in\mathbb{N})$ would be in the domain of the Dirichlet space locally. We emphasize that no Dirichlet spaces associated with the (fully) labeled diffusion $\mathbf{X}$ exist. So we instead introduce an infinite sequence of Dirichlet spaces associated with the k-labeld process $\{((X^1_t,\dots,X^k_t,\sum_{j>k}\delta_{X^j_t}))\}$ for all $k=0,1,\dots$. This sequence of the k-labeld processes have some the consistency and satisfies the SDEs (2) and (4).

Let μ be a probability measure on $(\mathsf{S},\mathcal{B}(\mathsf{S}))$. Let ρ^k be the k-correlation function of μ with respect to the Lebesgue measure. Let μ^k be the measure on $S^k\times\mathsf{S}$ defined by

$$\mu^k(A\times B)=\int_A \mu_{\mathbf{x}}(B)\rho^k(\mathbf{x})d\mathbf{x}. \tag{6}$$

Here $\mathbf{x}=(x_1,\dots,x_k)\in S^k$ and $d\mathbf{x}=dx_1\cdots dx_k$. Moreover, $\mu_{\mathbf{x}}$ is the Palm measure conditioned at $\mathbf{x}$:

$$\mu_{\mathbf{x}}=\mu(\cdot-\sum_{i=1}^k \delta_{x_i}|\ \mathsf{s}(x_i)\ge 1 \text{ for } i=1,\dots,k). \tag{7}$$

We now introduce Dirichlet forms describing the k-labeled dynamics. For a subset $A\subset S$ we define the map $\pi_A:\mathsf{S}\to\mathsf{S}$ by $\pi_A(\mathsf{s})=\mathsf{s}(A\cap\cdot)$. We say a function $f:\mathsf{S}\to\mathbb{R}$ is local if f is $\sigma[\pi_A]$-measurable for some compact set $A\subset S$. We say f is smooth if $\tilde{f}$ is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in (s_i) such that $f(\mathsf{s})=\tilde{f}((s_i))$ for $\mathsf{s}=\sum_i\delta_{s_i}$.

Let $\mathcal{D}_\circ$ be the set of all local, smooth functions on S with compact support. For $f,g\in\mathcal{D}_\circ$ we set $\mathbb{D}^{\mathbf{a}}[f,g]:\mathsf{S}\to\mathbb{R}$ by

$$\mathbb{D}^{\mathbf{a}}[f,g](\mathsf{s})=\frac{1}{2}\sum_i\sum_{m,n=1}^d \mathbf{a}_{mn}(s_i,\mathsf{s}_i^*)\frac{\partial\tilde{f}}{\partial s_{im}}\frac{\partial\tilde{g}}{\partial s_{in}}. \tag{8}$$

Here $\mathsf{s}=\sum_i\delta_{s_i}$, $\mathsf{s}_i^*=\sum_{j\ne i}\delta_{s_j}$, and $s_i=(s_{i1},\dots,s_{id})\in S$.

For $k\in\mathbb{N}$ let $\mathcal{D}_\circ^k=C_0^\infty(S^k)\otimes\mathcal{D}_\circ$. For $f,g\in\mathcal{D}_\circ^k$ let $\nabla^{\mathbf{a},k}[f,g]$ be such that

$$\nabla^{\mathbf{a},k}[f,g](\mathbf{x},\mathsf{s})=\frac{1}{2}\sum_{i=1}^k\sum_{m,n=1}^d \mathbf{a}_{mn}\frac{\partial f(\mathbf{x},\mathsf{s})}{\partial x_{im}}\frac{\partial g(\mathbf{x},\mathsf{s})}{\partial x_{in}}. \tag{9}$$

where $\mathbf{x}=(x_1,\dots,x_k)\in S^k$ and $x_i=(x_{i1},\dots,x_{id})\in S$. We set $\mathbb{D}^{\mathbf{a},k}$ by

$$\mathbb{D}^{\mathbf{a},k}[f,g](\mathbf{x},\mathsf{s})=\nabla^{\mathbf{a},k}[f,g](\mathbf{x},\mathsf{s})+\mathbb{D}^{\mathbf{a}}[f(\mathbf{x},\cdot),g(\mathbf{x},\cdot)](\mathsf{s}). \tag{10}$$

Let $L^2(\mu^k)=L^2(S^k\times\mathsf{S},\mu^k)$. Let $(\mathcal{E}^{\mathbf{a},k},\mathcal{D}_\circ^{\mathbf{a},k})$ be the bilinear form defined by

$$\mathcal{E}^{\mathbf{a},k}(f,g)=\int_{S^k\times\mathsf{S}}\mathbb{D}^{\mathbf{a},k}[f,g]d\mu^k,\quad \mathcal{D}_\circ^{\mathbf{a},k}=\{f\in\mathcal{D}_\circ^k\cap L^2(\mu^k);\ \mathcal{E}^{\mathbf{a},k}(f,f)<\infty\}. \tag{11}$$

We assume there exists a probability measure μ on S satisfying (A.1)–(A.5):

(A.1) ρ^k is locally bounded for each $k\in\mathbb{N}$.

(A.2) There exists $\mathbf{d}^\mu=(\mathbf{d}_m^\mu)_{m=1,\dots,d}\in\{L^1_{\mathrm{loc}}(\mu^1)\}^d$ such that

$$\int_{S\times\mathsf{S}}\mathbf{d}^\mu f d\mu^1=-\int_{S\times\mathsf{S}}\nabla_x f d\mu^1\quad\text{for all } f\in\mathcal{D}_\circ^1. \tag{12}$$

Here $\nabla_x f(x,\mathsf{s})=(\frac{\partial f(x,\mathsf{s})}{\partial x_m})_{m=1,\dots,d}$, where $x=(x_1,\dots,x_d)$. Moreover, the column vector $\mathbf{d}^\mu$ satisfies

$$\mathbf{b}=\frac{1}{2}\{\nabla_x\mathbf{a}\}\mathbf{d}^\mu+\frac{1}{2}\mathbf{a}\mathbf{d}^\mu,\quad \mathbf{b}\in L^2_{\mathsf{loc}}(\mu^1). \tag{13}$$

Here $\nabla_x\mathbf{a}$ is the matrix defined by $\nabla_x\mathbf{a}=[\frac{\partial\mathbf{a}_{mn}(x,\mathsf{s})}{\partial x_n}]$.

(A.3) $(\mathcal{E}^{\mathbf{a},k},\mathcal{D}_\circ^{\mathbf{a},k})$ is closable on $L^2(\mu^k)$ for each $k\in\{0\}\cup\mathbb{N}$.

(A.4) $\mathrm{Cap}^\mu(\{\mathsf{S}_{\mathrm{s.i.}}\}^c)=0$.

(A.5) There exists $T>0$ such that for each $R>0$

$$\liminf_{r\to\infty}\{\int_{|x|\le r+R}\rho^1(x)dx\}\cdot\ell(\frac{r}{\sqrt{(r+R)T}})=0,\qquad\text{where } \ell(t)=(2\pi)^{-1/2}\int_t^\infty e^{-u^2/2}du. \tag{14}$$

Let $(\mathcal{E}^{\mathbf{a},k}, \mathcal{D}^{\mathbf{a},k})$ be the closure of $(\mathcal{E}^{\mathbf{a},k}, \mathcal{D}_{\circ}^{\mathbf{a},k})$ on $L^2(\mu^k)$. It is known that $(\mathcal{E}^{\mathbf{a},k}, \mathcal{D}^{\mathbf{a},k})$ is quasi-regular and the associated diffusion exists. Cap^{μ} in (A.4) is the capacity of the Dirichlet space $(\mathcal{E}^{a,0}, \mathcal{D}^{a,0}, L^2(\mu))$.

Theorem 1. *Assume* (A.1)–(A.5). *Then there exists a set* $\mathsf{S}_0 \in \mathcal{B}(\mathsf{S})$ *such that*

$$\mu(\mathsf{S}_0) = 1, \quad \mathsf{S}_0 \subset \mathsf{S}_{\mathrm{s.i.}}, \tag{15}$$

and that, for all $\mathbf{s} \in \kappa^{-1}(\mathsf{S}_0)$*, there exists a* $S^{\mathbb{N}}$*-valued continuous process* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$*, and* $\mathbb{R}^{\mathbb{N}}$*-valued Brownian motion* $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i*})dB_t^i + \mathbf{b}(X_t^i, \mathsf{X}_t^{i*})dt \quad (i \in \mathbb{N}) \tag{16}$$

$$\mathbf{X}_0 = \mathbf{s}. \tag{17}$$

Moreover, $\mathbf{X}$ *satisfies*

$$P(\kappa(\mathbf{X}_t) \in \mathsf{S}_0,\ 0 \le \forall t < \infty) = 1, \tag{18}$$

$$P(\sup_{0\le t\le u} |X_t^i| < \infty \text{ for all } u \in \mathbb{N}, i \in \mathbb{N}) = 1. \tag{19}$$

Let $\kappa : S^{\mathbb{N}} \to \mathsf{S}$ such that $\kappa((s_i)) = \sum_i \delta_{s_i}$. Let $\kappa_{\mathrm{path}} : C([0,\infty); S^{\mathbb{N}}) \to C([0,\infty); \mathsf{S})$ such that $\kappa_{\mathrm{path}}(\mathbf{X}) = \sum_i \delta_{X_t^i}$.

Theorem 2. (1) *Let* $\mathbf{S}_0$ *be the subset of* $S^{\mathbb{N}}$ *defined by* $\mathbf{S}_0 = \kappa^{-1}(\mathsf{S}_0)$*. Let* $\mathbf{P}_{\mathbf{s}}$ *be the distribution of* $\mathbf{X}$ *given by Theorem 1. Then* $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$ *is a diffusion with state space* $\mathbf{S}_0$*.*
(2) *Let* $\mathsf{s} = \kappa(\mathbf{s})$*. Let* P_{s} *be the distribution of* $\mathsf{X} := \kappa_{\mathrm{path}}(\mathbf{X})$*. Then* $\{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{S}_0}$ *is a* μ*-reversible diffusion with state space* S_0*.*

Example 1. Let $\Psi(x,y)$ be a Ruelle's class potential that is smooth on $\{x \neq y\}$. Let μ_Ψ be the associated canonical Gibbs measures. Then (A.1)–(A.3) are satisfied. The sutablity of (A.4) and (A.5) is easily checked.

Example 2. Let Ψ be the 2D Coulomb potential $\Psi(x) = -2\log|x|$ $(x \in \mathbb{R}^2)$ with $\beta = 2$. Let $d = 2$. Then the associated SDE becomes

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad (\mathbf{X}_0 = (x_i)_{i\in\mathbb{N}}). \tag{20}$$

Theorem 3. *Let* μ *be the Ginibre random point field. Then there exists a set* $\mathbf{S} \subset (\mathbb{R}^2)^{\mathbb{N}}$ *such that* $\mu(\{\sum_{i\in\mathbb{N}} \delta_{x_i}; \mathbf{x} = (x_i) \in \mathbf{S}\}) = 1$ *and that* (20) *has a solution for all initial points* $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \mathbf{S}$*. Moreover, for all initial points* $\mathbf{x} \in \mathbf{S}$*,*

$$P(\mathbf{X}_t \in \mathbf{S} \cap \mathbf{S}_{\mathrm{single}} \text{ for all } t) = 1.$$

Here $\mathbf{S}_{\mathrm{single}} = \{\mathbf{s} = (s_i) \in (\mathbb{R}^2)^{\mathbb{N}};\ s_i \neq s_j \text{ if } i \neq j\}$*. More precisely, there exist* $(\mathbb{R}^2)^{\mathbb{N}}$*-valued process* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *and Brownian motion* $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ *such that the pair* $(\mathbf{X}, \mathbf{B})$ *satisfies the SDE* (20).

We remark that the DLR equation for μ does not make sense. However, by (20) one can say μ is a measure with $2D$ Coulomb interaction potential Ψ. Indeed, μ is the reversible measure of the unlabeled diffusion $\mathbb{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$, where the associated labeled dynamics $\mathbf{X}_t = (X_t^i) \in (\mathbb{R}^2)^{\mathbb{N}}$ is the solution of the infinitely dimensional SDE:

The Ginibre random point field μ is a probability measure on the configuration $\mathbb{S}$ over $\mathbb{R}^2$. It is known that μ is translation and rotation invariant. Moreover, μ is so called a determinantal random point field whose n-correlation function ρ^n is given by

$$\rho^n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n}, \tag{21}$$

where $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ is the kernel defined by

$$K(x, y) = \frac{1}{\pi} \exp(-\frac{|x|^2}{2} - \frac{|y|^2}{2}) \cdot e^{x\bar{y}}. \tag{22}$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence: $\mathbb{R}^2 \ni x = (x_1, x_2) \mapsto x_1 + \sqrt{-1}x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \sqrt{-1}y_2$ means the complex conjugate under this identification.

The key point of Theorem 3 is to calculate the log derivative of the one moment measure μ^1 of the Ginibre random point field μ.

The solution satisfies the second SDE:

Theorem 4. *For each* $\mathbf{s} \in \mathbf{S}$, $(\mathbf{X}, \mathbf{B})$ *in Theorem 3 satisfies*

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r \to \infty} \sum_{|X_t^j| < r,\, j \ne i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}) \tag{23}$$

$$\mathbf{X}_0 = \mathbf{s}. \tag{24}$$

干渉ポテンシャル Ψ をもつ、干渉ブラウン運動について、つぎの 3 つの基本的な問題が、まだ十分に解決されていない。以下の問題はすべて、Ruelle クラスポテンシャルに話を限っても、Lennard-Jones 6-12 ポテンシャルのように代表なものに対して証明されていない。

(A) SDE の強解の存在と一意性　（上述の定理で得た解は、ブラウン運動だけの関数ではない）

(B) SDE が解を持ち実際にに運動する部分集合の「明示的な」特徴付け。つまり非平衡問題。(上述の結果は、capacity ゼロの集合が、何になるかが明示的にはわからない)

(C) 各粒子が爆発しないという条件 (A.5) の下での、Dirichlet 形式の拡張の一意性：

この問題について、以下の結果が知られている。

• (A) に対しては、$\Psi \in C_0^3(\mathbb{R}^d)$ (Lang, Fritz)、また、Ψ がハードコアをもち、遠方で指数的に減少するという条件 （種村、種村-Roelly）という 2 種類の結果しかない。

• (B) に対しては、$\Psi \in C_0^3(\mathbb{R}^d)$ かつ次元 d が 4 次元以下 (Fritz)。1 次元（Rost など)。

• (C) に対しては、ハードコア・ブラウンボール (種村)。

この講演では、Dirichlet 形式の方法によって、(A) をいかに解決するか、という問題について、可能性のあると思われる、一つのアイデアを説明する。

参考文献

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