Laplace approximation for rough differential equations driven by fractional Brownian motion

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In this talk, we prove the Laplace-type asymptotics for the solution of a rough differential equation driven by (the lift of) fractional Brownian motion of the Hurst parameter H $(1/4 < H \leq 1/2)$. This is an "FBM version" of the well-known result for SDEs driven by the usual Brownian motion. In this talk. (stochastic or ordinary) differential equations are understood in the sense of the rough path theory. Unlike the BM case (i.e., $H = 1/2$), the third level paths (the triple integrals) of FBM also play a role when $1/4 < H \leq 1/3$.

A real-valued continuous stochastic process $(w^H_t)_{t\geq 0}$ starting at 0 is said to a fBm of Hurst parameter H if it is a centered Gaussian process with

$$\mathbb{E}[w^H_t w^H_s] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \qquad (s,t \geq 0)$$

This process has stationary increments $\mathbb{E}[(w^H_t - w^H_s)^2] = |t-s|^{2H}$ $(s,t \geq 0)$, and the scaling properity, i.e., for any $c > 0$, $(c^{-H} w^H_{ct})_{t\geq 0}$ and $(w^H_t)_{t\geq 0}$ have the same law. Note that $(w^{1/2}_t)_{t\geq 0}$ is the standard Brownian motion. For $d \geq 1$, a d-dimensional fBm is defined by $(w^{H,1}_t, \ldots, w^{H,d}_t)_{t\geq 0}$, where $w^{H,i}$ $(i = 1,\ldots,d)$ are independent one-dimensional fBm's. Its law μ^H is a probability measure on $C_0([0,1], \mathbf{R}^d)$. (Actually, it is a measure on $C_0^{p-var}([0,1], \mathbf{R}^d)$ for $p > 1/H$, or on $C_0^{\alpha-hldr}([0,1], \mathbf{R}^d)$ for $\alpha < H$).

For $2 < p < 4$, let $G\Omega_p(\mathbf{R}^d)$ denotes the geometric rough path space. A $\mathbf{R}^d$-valued finite variational path $x \in C_0^{1-var}([0,1], \mathbf{R}^d)$ is naturally lifted as an element of $G\Omega_p(\mathbf{R}^d)$ by the followiing iterated Stieltjes integral;

$$X^j_{s,t} = \int_{s\leq t_1 \leq \cdots \leq t_j \leq t} dx_{t_1} \otimes dx_{t_2} \otimes \cdots \otimes dx_{t_j}. \qquad (1)$$

We say X is the smooth rough path lying above x. In a similar way, for $1 < q < 2$, $x \in C_0^{q-var}([0,1], \mathbf{R}^d)$ can naturally be lifted, where the iterated integral in (1) should be understood in the sense of Young.

Let $1/4 < H \leq 1/2$ and $1/H < p < [1/H] + 1$. By Coutin-Qian's result $W^H(m)$. i.e., the lift of the dyadic piecewise linear approximation $w^H(m)$ converges a.s. in $G\Omega_p(\mathbf{R}^d)$. We write $W^H := \lim_{m\to\infty} W^H(m)$ and call it fractional Brownian rough path. (It is not possible to show the existence of W^H for $0 < H < 1/4$ with their method. In a framework different from the original one of T. Lyons, Tindel and Unterberger recently showed existence of the lift of w^H for any H. This "algebraic" framework was proposed by M. Gubinelli and might be interesting.)

In this talk, we consider the following RDE; for $\varepsilon > 0$,

$$dY^\varepsilon_t = \sigma(Y^\varepsilon_t)\varepsilon dW^H_t + \beta(\varepsilon, Y^\varepsilon_t)dt, \qquad Y^\varepsilon_0 = 0. \qquad (2)$$

Here, $\sigma \in C_b^\infty(\mathbf{R}^n, \mathrm{Mat}(n,d))$ and $\beta \in C_b^\infty([0,1] \times \mathbf{R}^n, \mathbf{R}^n)$. Note that C_b^∞ denotes the set of bounded smooth functions with bounded derivatives. Note also that Y^ε is a $G\Omega_p(\mathbf{R}^n)$-valued random variable.

Let $\mathcal{H}^H$ be the Cameron-Martin subspace of the d-dimensional fBm $(w^H_t)_{0\leq t\leq 1}$. By Friz-Victoir's result, $k \in \mathcal{H}^H$ is of finite q-variation for any $(H + 1/2)^{-1} < q < 2$. Hence, the following ODE makes sense in the q-variational setting in the sense of the Young integration;

$$dy_t = \sigma(y_t)dk_t + \beta(0, y_t)dt, \qquad y_0 = 0.$$

Note that y is again of finite q-variation and we will write $y = \Psi(k)$.

Now we set the following assumptions. In short, we assume that there is only one point that attains minimum of F_Λ and the Hessian at the point is non-degenerate. These are typical assumptions for Laplace's method of this kind. The space of continuous paths in $\mathbf{R}^n$ with finite p-variation starting at 0 is denoted by $C_0^{p-var}([0,1], \mathbf{R}^n)$. Note that the self-adjoint operator A in the fourth assumption turns out to be Hilbert-Schmidt.

(H1): F and G are real-valued bounded continuous function on $C_0^{p-var}([0,1], \mathbf{R}^n)$ for some $p > 1/H$.

(H2): The function $F_\Lambda := F \circ \Psi + \| \cdot \|^2_{\mathcal{H}^H}/2$ attains its minimum at a unique point $\gamma \in \mathcal{H}^H$. We will write $\phi^0 = \Psi(\gamma)$.

(H3): F and G are $m+3$ and $m+1$ times Fréchet differentiable on a neighborhood $U(\phi^0)$ of $\phi^0 \in C_0^{p-var}([0,1], \mathbf{R}^n)$, respectively. Moreover, there are positive constants $M_1, M_2, \dots$ such that

$$
\begin{aligned}
|\nabla^j F(\eta)\langle z, \dots, z\rangle| &\leq M_j \|z\|^j_{p'-var}, \qquad (j = 1, \dots, m+3) \\
|\nabla^j G(\eta)\langle z, \dots, z\rangle| &\leq M_j \|z\|^j_{p'-var}, \qquad (j = 1, \dots, m+1)
\end{aligned}
$$

hold for any $\eta \in U(\phi^0)$ and $z \in C_0^{p-var}([0,1], \mathbf{R}^n)$.

(H4): At the point $\gamma \in \mathcal{H}^H$, the bounded self-adjoint operator A on $\mathcal{H}^H$, which corresponds to the Hessian $\nabla^2(F \circ \Psi)(\gamma)|_{\mathcal{H}^H \times \mathcal{H}^H}$, is strictly larger than $-\mathrm{Id}_{\mathcal{H}^H}$ (in the form sense).

Now we state our main theorem. Under these assumptions, the following Laplace-type asymptotics holds. (Below, $Y^{\varepsilon,1} = (Y^\varepsilon)^1$ denotes the first level path of Y^ε);

Theorem 1 *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ and $\beta : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞. Then, under Assumtions* **(H1)** *–* **(H4)***, we have the following asymptotic expansion as $\varepsilon \searrow 0$; there are real constants c and $\alpha_0, \alpha_1, \dots$ such that*

$$
\begin{aligned}
&\mathbb{E}\big[G(Y^{\varepsilon,1}) \exp\big(-F(Y^{\varepsilon,1})/\varepsilon^2\big)\big] \\
&\quad = \exp(-F_\Lambda(\gamma)/\varepsilon^2) \exp(-c/\varepsilon) \cdot \big(\alpha_0 + \alpha_1 \varepsilon + \cdots + \alpha_m \varepsilon^m + O(\varepsilon^{m+1})\big).
\end{aligned}
$$

The proof is similar to the one for Brownian rough path (i.e., the case $H = 1/2$). The following facts are the keys; (i) A Fernique-type theorem for W^H. (ii) A Cameron-Martin-type for W^H. (iii) Taylor expansion for the Itô map or RDE (2) around the minimum point $\gamma \in \mathcal{H}^H$. However, (iii) was done in the speaker's previous paper.

For those who understand the proof for Brownian rough path, the most difficult part is perhaps how to treat elements of the Cameron-Martin space $\mathcal{H}^H$, in particular, the proof of the Hilbert-Schmidt property of the Hessian A. Thanks to Friz-Victoir's result, those Cameron-Martin paths are of finite q-variation for some $1 < q < 2$ such that $1/p + 1/q > 1$. Thus, we can use Young integration theory.

Consider the short time problem for the law of V_t, which is a unique solution of the following RDE;

$$
dV_t = \sigma(V_t) dW_t^H + b(V_t) dt, \qquad Y_0 = 0.
$$

Here, $b : \mathbf{R}^n \to \mathbf{R}^n$ is C_b^∞, which is independent of small parameter ε this time. By the scaling property of fractional Brownian rough path, the problem reduces to studying following RDE;

$$
dY_t^\varepsilon = \sigma(Y_t^\varepsilon) \varepsilon dW_t^H + \varepsilon^{1/H} b(Y_t^\varepsilon) dt, \qquad Y_0^\varepsilon = 0
$$

Although fractional power of ε is involved, we can show that Theorem 1 above also holds for this case since $1/H \geq 1/2$. As a result, under certain mild assumptions, we can prove the Laplace-type asymptotics for the law of V_t as $t \searrow 0$.