Spectral analysis of relativistic Schrödinger operators by path measures

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1 Relativistic Schrödinger operators

Relativistic Schrödinger operator with vector potential a is defined formally by

$$H = \sqrt{(p-a)^2 + m^2} - m + V,$$

where $p = -i\nabla$, m denotes the mass of electron, $a = (a_1, ..., a_d)$ vector potentials and V an external potential. Let us suppose that $a \in (L^2_{loc}(\mathbb{R}^d))^d$. Then the kinetic term $\frac{1}{2}(p-a)^2$ can be defined through the quadratic form

$$(f, g) \mapsto \frac{1}{2}\sum_{\mu=1}^{d}((p_\mu - a_\mu)f, (p_\mu - a_\mu)g).$$

The self-adjoint operator associated with this quadratic form is denoted by h. Under the assumption $0 \le V_+ \in L^1_{loc}(\mathbb{R}^d)$ and $0 \le V_-$ is relatively form bounded with respect to $(1/2)p^2$, Then the relativistic Schrödinger operator is rigorously defined as a self-adjoint operator on $L^2(\mathbb{R}^d)$ by

$$H = (2h + m^2)^{1/2} - m \dot{+} V_+ \dot{-} V_-.$$

Her $\dot{\pm}$ is the quadratic form sum. It can be seen that $C_0^\infty(\mathbb{R}^d)$ is a form core of H. Let $(T_t)_{t\ge 0}$ be the subordinator such that $\mathbb{E}[e^{-uT_t}] = e^{-t(\sqrt{2u+m^2}-m)}$. In addition to condistions on a and V mentioned above we furthermore suppose that $\nabla \cdot a \in L^1_{loc}(\mathbb{R}^d)$. Then by using the Brownian motion $(B_t)_{t\ge 0}$ independent of the subordinator the path integral representation of $(f, e^{-tH}g)$ is given by the theorem:

Theorem 1.1

$$(f, e^{-tH}g) = \int dx \mathbb{E}^{x,0}\left[\overline{f(B_{T_0})} g(B_{T_t}) e^{S_t}\right],$$

where the exponent S_t is given by $-\int_0^t V(B_{T_s})ds - i\int_0^{T_t} a(B_s) \circ dB_s$.

From this path integral representation we can immediately see that e^{-tH} is ultarcontractive, i.e., e^{-tH} maps L^p to L^q for all $1 \le p \le q \le \infty$ for Kato-class potential. This procedure includes not only relativistic Schrödinger operators, but also Schrödinger operator with Bernstein function of the Laplacian, i.e. $\Psi(h) + V$ for any Bernstein function Ψ such that $\Psi(0) = 0$.

2 QFT version

The Pauli-Fierz model is a model in the so-called *nonrelativistic QED*. This model can be extended to a relativistic one. This model is defined on $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}$, where $\mathscr{F}$ is a boson Fock space. Define

$$H_P = \sqrt{(p \otimes 1 - \alpha A)^2 + m^2} - m + V \otimes 1 + 1 \otimes H_{\mathrm{rad}},$$

where $\alpha \in \mathbb{R}$ is a coupling constant, A denotes the quantized radiation field given by $A_\mu = \int^\oplus A_\mu(x)dx$ under the identification $\mathscr{H} = \int^\oplus \mathscr{F} dx$ and $A_\mu(x)$ by

$$A_\mu(x) = \sum_{j=1}^{d-1} \int \frac{\hat{\varphi}(k)}{|k|} e_\mu(k,j) \left(a^\dagger(k,j) e^{-ikx} + a(k,j) e^{+ikx} \right) dk.$$

$a^\dagger$ and a satisfy canonical commutation relations $[a(k,j), a^\dagger(k',j')] = \delta_{jj'}\delta(k-k')$ and $\{e(k,1),...,e(k,d-1),k/|k|\}$ forms an orthogonal base on the tangent space of the $d-1$-dimensional unit sphere at k, $T_k S_{d-1}$. H_{rad} is the free Hamiltonian defined by $H_{\mathrm{rad}} = \sum_{j=1}^{d-1} \int |k| a^\dagger(k,j) a(k,j) dk$. In the case of $\alpha = 0$ the Hamiltonian is

$$(\sqrt{p^2 + m^2} - m + V) \otimes 1 + 1 \otimes H_{\mathrm{rad}}$$

and all the eigenvalues of $\sqrt{p^2+m^2} - m + V$ are embedded in the continuous spectrum since $\sigma(H_{\mathrm{rad}}) = [0,\infty)$. Thus to investigate the spectrum of H_P but with $\alpha \neq 0$ is a difficult issue. The boson Fock space is identified with the probability space $L^2(\mathscr{M}, \mu_0)$ with $\mathscr{M} = \oplus^d \mathscr{S}'(\mathbb{R}^d)$ endowed with a certain Gaussian measure μ such that $\mathbb{E}[\mathscr{A}_\mu(f)\mathscr{A}_\nu(g)] = \frac{1}{2}\int \bar{\hat{f}}(k)\hat{g}(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right) dk$. We can construct the functional integral representation of $(F, e^{-tH_P}G)$.

Theorem 2.1

$$(F, e^{-tH_P}G) = \int dx \mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s})ds} \int_{\mathscr{E}} \overline{F(\mathscr{A}_0, B_{T_0})} G(\mathscr{A}_t, B_{T_t}) e^{-iK_t} d\mu \right], \quad F, G \in \mathscr{H}.$$

Here $\mathscr{E}$ is the Euclidean version of $\mathscr{M}$ and $\mathscr{A}_t$ is the Euclidean field with time t. The exponent is of the form $K_t = \int_0^t \mathscr{A}_s(\tilde{\varphi}(\cdot - B_s)) \cdot dB_s$, where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}/|k|$.

By means of this functional integral representation we can show that

1 H_P is self-adjoint on $D(\sqrt{p^2} \otimes 1) \cap D(1 \otimes H_{\mathrm{rad}})$;

2 $e^{-i(\pi/2)N} e^{-tH_P} e^{i(\pi/2)N}$ is a positivity improving operator, where N denotes the number operator ;

3 the ground state of H_P is unique;

4 the ground state is spatially exponentially decay for $m > 0$.

These results can be extended to more general models of the form:

$$H_\Psi = \Psi\left(\frac{1}{2}(p \otimes 1 - \alpha A)^2\right) + V \otimes 1 + 1 \otimes H_{\mathrm{rad}}$$

with an arbitrary Bernstein functions.