Malliavin-Thalmaier公式による多次元密度関数の推定とファイナンスへの応用

Kazuhiro Yasuda [1] (joint work with A. Kohatsu-Higa)

Abstract. The Malliavin-Thalmaier formula was introduced in [2] for use in Monte-Carlo simulation. This is an integration by parts formula for high dimensional probability density functions. But when this formula is applied directly for computer simulation, we show that it is unstable. We propose an approximation to the Malliavin-Thalmaier formula. In the first part, we give the central limit theorem to obtain the values of the parameters in Monte-Carlo simulations which achieves a prescribed error level. To prove it, we need the order of the bias and the variance of the approximation error, and we prove the central limit theorem by using these error estimation. And in the latter part, we obtain an explicit Malliavin-Thalmaier formula for the calculation of Greeks in finance. The weights obtained are free from the curse of dimensionality.

1 CLT for the Approximated Malliavin-Thalmaier Formula

We give the rate of convergence of the modified estimator of the density at $\mathbf{x} \in \mathbb{R}^d$.

Definitions and Notations

1. For $h > 0$ and $\mathbf{x} \in \mathbb{R}^d$, define $|\cdot|_h$ by $|\mathbf{x}|_h := \sqrt{\sum_{i=1}^d x_i^2 + h}$. Without loss of generality, we assume $0 < h < 1$.

2. For $i = 1, ..., d$, define the approximated once derivative of the fundamental solution of Poisson equation; for $\mathbf{x} \in \mathbb{R}^d$, $\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}$, where A_d is some constant.

3. Then we define the approximation to the density function of F as; for $\mathbf{x} \in \mathbb{R}^d$,

$$p_F^h(\mathbf{x}) := E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x}) H_{(i)}(F; 1)\right],$$

where $H_{(i)}(F;1) := \sum_{j=1}^d D^*((\gamma_F^{-1})^{ij} DF^j)$ for $i = 1, ..., d$. This is an approximation of the Malliavin-Thalmaier formula.

In what follows $\Rightarrow$ denotes weak convergence and the index $j = 1, ..., N$ denote N independent copies of the respective random variables.

Theorem 1.1 *Let Z be a random variable with standard normal distribution. And $F^{(j)} \in (\mathbb{D}^\infty)^d$ is a random vector which has independent identical distribution.*

(i). *When $d = 2$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^2 \ln \frac{1}{h}}$ for some positive constant C fixed throughout.*

$$n\left(\frac{1}{N}\sum_{j=1}^N \sum_{i=1}^2 \frac{\partial}{\partial x_i} Q_2^h(F^{(j)} - \mathbf{x}) H_{(i)}(F;1)^{(j)} - p_F(\mathbf{x})\right) \Longrightarrow \sqrt{C_3^{\mathbf{x}}} Z - C_1^{\mathbf{x}} C,$$

where $H_{(i)}(F;1)^{(j)}$, $i = 1, ..., d$, $j = 1, ..., N$, denotes the weight obtained in the j-th independent simulation (the same that generates $F^{(j)}$) and $C_1^{\mathbf{x}}$, $C_3^{\mathbf{x}}$ are some constants.

(ii). *When $d \geq 3$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^{\frac{d}{2}+1}(\ln \frac{1}{h})^2}$ for some positive constant C fixed throughout.*

$$n\left(\frac{1}{N}\sum_{j=1}^N \sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F^{(j)} - \mathbf{x}) H_{(i)}(F;1)^{(j)} - p_F(\mathbf{x})\right) \Longrightarrow \sqrt{C_4^{\mathbf{x}}} Z - C_1^{\mathbf{x}} C,$$

[1] Graduate school of Engineering Science, Osaka University. e-mail: yasuda@sigmath.es.osaka-u.ac.jp

where $C_1^{\mathbf{x}}$, $C_4^{\mathbf{x}}$ are some constants.

To prove Theorem 1.1, we need the order of the error of the approximation to the density.

Proposition 1.2 *Let F be a nondegenerate random vector, then for $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$,*

$$p_F(\mathbf{x}) - p_F^h(\mathbf{x}) = C_1^{\mathbf{x}} h \ln \frac{1}{h} + C_2^{\mathbf{x}} h + o(h),$$

where $C_1^{\mathbf{x}}$ and $C_2^{\mathbf{x}}$ are constants which depend on $\mathbf{x}$, but are independent of h. The constants can be written explicitly.

Proposition 1.3 *Let F be a nondegenerate random vector. For $\mathbf{x} \in \mathbb{R}^d$,*

$$E\left[\left(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d^h(F-\mathbf{x}) H_{(i)}(F;1) - p_F(\mathbf{x})\right)^2\right] = \begin{cases} C_3^{\mathbf{x}} \ln \frac{1}{h} + O(1) & d = 2, \\ C_4^{\mathbf{x}} \frac{1}{h^{\frac{d}{2}-1}} + o\left(\frac{1}{h^{\frac{d}{2}-1}}\right) & d \geq 3 \end{cases}$$

where $C_3^{\mathbf{x}}$ and $C_4^{\mathbf{x}}$ are constants which depend on $\mathbf{x}$, but is independent of h. The constants can be written explicitly.

2 Application to Greeks Calculation

We compute Greeks using the Malliavin-Thalmaier Formula. We consider a random vector $F^\mu = (F_1^\mu, ..., F_d^\mu)$, $\mu \in \mathbb{R}^m$, $m \in \mathbb{N}$ which depends on a parameter μ. Suppose that $F^\mu \in (\mathbb{D}^\infty)^d$ is a nondegenerate random vector. And let $f(x_1, ..., x_d)$ be a payoff function in a class [2]

$$\mathcal{A} := \left\{ f : \mathbb{R}^d \to \mathbb{R} : \begin{array}{l} \text{continuous a.e. wrt Lebesgue measure, and} \\ \text{there exist constants } c, a \text{ such that } |f(\mathbf{x})| \leq \frac{c}{(1+|\mathbf{x}|)^{ad}} \ (a > 1) \end{array} \right\}.$$

We denote the integration with respect to $p_{F^\mu,G}^h(\mathbf{x})$ by $E^h[\cdot]$. That is, $E^h\left[f(F^\mu)\right] := \int\cdots\int_{\mathbb{R}^d} f(\mathbf{x}) p_{F^\mu,1}^h(\mathbf{x}) d\mathbf{x}$. And for $i, j = 1, ..., d$, set $g_{i,j}^h(\mathbf{y}) := \frac{\partial}{\partial y_j} \int\cdots\int_{\mathbb{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q_d^h(\mathbf{y}-\mathbf{x}) d\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$.

Theorem 2.1 *Let $k \in \{1, ..., m\}$ be fixed. Let $f \in \mathcal{A}$. Let F^μ be a nondegenerate random vector, which is differentiable with respect to μ_k with $\frac{\partial F_j^\mu}{\partial \mu_k} \in \mathbb{D}^\infty$ for $j = 1, ..., d$. We assume that for all $i = 1, ..., d$, there exists some $g_{i,i}$ such that $g_{i,i}^h \to g_{i,i}$ a.e. as $h \to 0$. Then as $h \to 0$,*

$$\frac{\partial}{\partial \mu_k} E^h\Big[f(F^\mu)\Big] = \sum_{i,j=1}^{d} E\left[g_{i,i}^h(F^\mu) H_{(j)}\left(F^\mu; \frac{\partial F_j^\mu}{\partial \mu_k}\right)\right] \longrightarrow \sum_{i,j=1}^{d} E\left[g_{i,i}(F^\mu) H_{(j)}\left(F^\mu; \frac{\partial F_j^\mu}{\partial \mu_k}\right)\right] = \frac{\partial}{\partial \mu_k} E\Big[f(F^\mu)\Big].$$

Remark 2.2 *We remark that in Theorem 2.1, $H_{(i)}$ requires only one Skorohod integral. Even if higher derivatives with respect to μ are considered this fact remains unchanged.*

References

[1] A. Kohatsu-Higa, K. Yasuda, *Estimating Multidimensional Density Functions using the Malliavin-Thalmaier Formula*, preprint.

[2] P. Malliavin, A. Thalmaier, *Stochastic calculus of variations in mathematical finance*, Springer Finance, Springer-Verlag, Berlin, 2006.

[2] Note that since stock price does not take negative value, a put option and a digital put option are in $\mathcal{A}$. In a digital call option case, we can transform as it follows; $\mathbf{1}_{[K,\infty)}(x) = 1 - \mathbf{1}_{[0,K)}(x)$. Finally in the case of a call option, a localization is needed.