A generalized local limit theorem for mixing semi-flows

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We consider a continuous dynamical system $S_t : M \to M$, $t \geq 0$, preserving a probability measure μ on a compact manifold M and a real-valued Borel measurable function $g : M \to \mathbb{R}$. Then the family $\{g \circ S_t; t \geq 0\}$ gives a stationary process with (M, μ) as the underlying probability space. An important problem in ergodic theory is whether this process satisfies the limit theorems such as the strong law of large numbers, the central limit theorem and the law of iterated logarithm. In this paper, we study the local limit theorem for such a stationary process to measure the dependence between the values of g at time 0 and time t (see [B] for the local limit theorem for independent random variables in the case of discrete time.).

We say that the stationary process $\{g \circ S_t; t \geq 0\}$ with $\int_M g d\mu = 0$ satisfies the *local limit theorem* if

$$\lim_{t\to\infty} \sup_{z\in\mathbb{R}} \left| \sqrt{t\sigma} \int_M u\left(z + \int_0^t g \circ S_\tau d\tau\right) d\mu - \frac{1}{\sqrt{2\pi\sigma}} e^{-z^2/2t\sigma} \int_{-\infty}^{\infty} u(\theta) d\theta \right| = 0 \tag{0.1}$$

for all rapidly decreasing functions u, where we assume the limit $\sigma := \lim_{t\to\infty} \frac{1}{t} \int_M (\int_0^t g \circ S_\tau d\tau)^2 d\mu$ exists and is not zero. This limit theorem implies that, for any finite interval I, we have

$$\left| \sqrt{t\sigma}\mu(\int_0^t g \circ S_\tau d\tau \in I + z) - m(I) \frac{1}{\sqrt{2\pi\sigma}} e^{-z^2/2t\sigma} \right| \to 0 \qquad \text{as } t \to \infty$$

uniformly for $z \in \mathbb{R}$, where m is the Lebesgue measure on $\mathbb{R}$.

Reference

[B] Breiman M., *Probability*. Adison-Wesley, 1968.