A stochasitic Taylor-like expansion in the rough path theory

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In this talk we establish a Taylor-like expansion in the context of the rough path theory for a family of Itô maps indexed by a small parameter. We treat not only the case that the roughness p satisfies $[p] = 2$, but also the case that $[p] \geq 3$. As an application, we discuss the Laplace asymptotics for Itô functionals of Brownian rough paths.

Let $\mathcal{V}, \mathcal{W}$ be real Banach spaces and let $X : [0,1] \to \mathcal{V}$ be a nice path in $\mathcal{V}$. Let us consider the following $\mathcal{W}$-valued ordinary differential equation (ODE);

$$dY_t = \sigma(Y_t)dX_t, \qquad \text{with } Y_0 = 0. \tag{1}$$

Here, σ is a nice function from $\mathcal{W}$ to the space $L(\mathcal{V}, \mathcal{W})$ of bounded linear maps. The correspondence $X \mapsto Y$ is called the Itô map and will be denoted by $Y = \Phi(X)$.

In the rough path theory of T. Lyons, the equation (1) is significantly generalized. First, the space of geometric rough paths on $\mathcal{V}$ with roughness $p \geq 1$, which contains all the nice paths, is introduced. It is denoted by $G\Omega_p(\mathcal{V})$ and its precise definition will be given in the next section. Then, the Itô map Φ extends to a continuous map from $G\Omega_p(\mathcal{V})$ to $G\Omega_p(\mathcal{W})$. In particular, when $2 < p < 3$ and $\dim(\mathcal{V}), \dim(\mathcal{W}) < \infty$, this equation (1) corresponds to a stratonovich-type stochastic differential equation (SDE). (See Lyons and Qian's book for the facts in this paragraph.)

In many fields of analysis it is quite important to investigate how the output of a map behaves asymptotically when the input is given small perturbation. The Taylor expansion in the calculus is a typical example. In this paper we investigate the behaviour of $\Phi(\varepsilon X + \Lambda)$ as $\varepsilon \searrow 0$ for a nice path Λ and $X \in G\Omega_p(\mathcal{V})$. Slightly generalizing it, we will consider the asymptotic behaviour of $Y^{(\varepsilon)}$, which is defined by (3) below, as $\varepsilon \searrow 0$;

$$dY_t^{(\varepsilon)} = \sigma(\varepsilon, Y_t^{(\varepsilon)})\varepsilon dX_t + b(\varepsilon, Y_t^{(\varepsilon)})d\Lambda_t, \qquad \text{with } Y_0^{\varepsilon} = 0. \tag{2}$$

Then, we will obtain an asymptotic expansion as follows; there exist $Y^0, Y^1, Y^2, \ldots$ such that

$$Y^{\varepsilon} \sim Y^0 + \varepsilon Y^1 + \cdots + \varepsilon^n Y^n + \cdots \qquad \text{as } \varepsilon \searrow 0.$$

We call it a stochastic Taylor-like expansion around a point Λ. Despite its name, this is purely real analysis and no probability measure is involved in the argument.

This kind of expansion in the context of the rough path theory was first done by Aida (2007) (for the case where coefficients σ, b are independent of ε, $[p] = 2$, $\mathcal{V}, \mathcal{W}$ are finite dimensional). Then, Inahama and Kawabi (2007) extended it to the infinite dimensional case in order to investigate the Laplace asymptotics for the Brownian motion over loop groups. (Those methods are slightly different. In [Aida (2007)], unlike in [IK (2007)], the derivative equation of the given equation is explicitly used.)

The main result in this paper is to generalize the stochastic Taylor-like expansion in [IK (2007)]. The following points are improved:

1. The roughness p satisfies $2 \leq p < \infty$. In other words, not only the case $[p] = 2$, but also the case $p \geq 3$ is discussed.

2. The coefficients σ and b depend on the small parameter $\varepsilon > 0$. In other words, we treat not just one fixed Itô map, but a family of Itô maps indexed by ε.

3. The base point Λ of the expansion is a continuous q-variational path for any $1 \leq q < 2$ with $1/p + 1/q > 1$. In [IK (2007)], Λ is a continuous bounded variational path (i.e., the case $q = 1$).

4. Not only estimates of the first level paths of $Y^0, Y^1, Y^2, \ldots$, but also estimates of the higher level paths are given.

The mothod we used to prove the expansion is basically the same as in [IK (2007)]. However, we must first generalize the local Lipschitz continuity of the integration map as the integrand varies. Put simply, the proposition states that the map

$$(f, X) \in C^{[p]+1}_{b,loc}(\mathcal{V}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \mapsto \int f(X)dX \in G\Omega_p(\mathcal{W})$$

is continuous. Here, $C^{[p]+1}_{b,loc}$ denotes the space of $[p]+1$-times Fréchet differentiable maps whose derivatives of order $0, 1, \dots, [p]+1$ are bounded on every bounded sets. Note that in [LQ], the integrand (or the coefficient of an ODE) is always fixed. In the path space analysis, a path on a manifold is often regarded as a current-valued path. This generalization is also necessary for such a viewpoint in the rough path context. Next, using the above fact, we must extend Lyons' continuity theorem (also known as the universal limit theorem) when the coefficient of the ODE varies. Put simply, the correspondence

$$(\sigma, X, y_0) \in C^{[p]+2}_{b,M}(\mathcal{W}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \times \mathcal{W} \mapsto Z = (X, Y) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$$

is continuous. Here, $Z = (X, Y)$ is the solution of (1) with the initial condition replaced with y_0 and $C^{[p]+2}_{b,M}(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ $(M > 0)$ is a subset of $C^{[p]+2}_{b}(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ (a precise definition is given later).

In the end, we discuss the Laplace asymptotics of Itô functionals of Brownian rough path, which is an improvement of [Aida(2007) ,IK(2007)]. Let $(\mathcal{V}, H, \mu)$ be an abstract Wiener space with a canonical $\mathcal{V}$-valued Brownian motion $(w_t)_{t\geq 0}$. We assume that $\mathcal{V} \otimes \mathcal{V}$ (the projective tensor product) and μ satisfy the "exactness" condition, which implies the existence of the Brownian rough path W.

For $\sigma \in C^{\infty}_b(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ and $b \in C^{\infty}_b([0, \infty) \times \mathcal{W}, \mathcal{W})$, we consider the following differential equation in the rough path sense:

$$dZ^{(\varepsilon)}_t = \sigma(Z^{(\varepsilon)}_t)\varepsilon dW_t + b(\varepsilon, Z^{(\varepsilon)}_t)dt, \qquad \text{with } Z^{\varepsilon}_0 = 0. \tag{3}$$

Let F, G be a continuous bounded functions on the continuous path space over $\mathcal{W}$. We assume that F, G are sufficiently regular. If we assume the traditional assumtions for F (existence of the unique minimum point and non-degeneracy of the Hessian there), we obtain the following theorem. This is a generalization of the main theorem of [IK (2007)].

Theorem 1 *We have the following asymptotic expansion:*

$$\begin{aligned} &\mathbb{E}\Big[G(Z^{\varepsilon}) \exp\big(-F(Z^{\varepsilon})/\varepsilon^2\big)\Big] \\ &= \exp\big(-F_{\Lambda}(\gamma)/\varepsilon^2 - c(\gamma)/\varepsilon\big) \cdot \big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_n\varepsilon^n + O(\varepsilon^{n+1})\big). \end{aligned} \tag{4}$$