Large deviation for stochastic line integrals as L^p-currents

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In this talk, we consider the large deviation principle for stochastic line integrals of Brownian paths on a compact Riemannian manifold. We regard them as a random map on a Sobolev space of vector fields. We show that the differentiability order of the Sobolev space can be chosen to be almost independent of the dimension of the underlying space under stronger integrability condition than L^2. The large deviation is formulated for the joint distribution of stochastic line integrals and the empirical distribution of a Brownian path. As the result, the rate function is given explicitly.

Let M be a d-dimensional closed Riemannian manifold with the normalized Riemannian measure m. Assume $d \geq 3$. Fix a constant $L > 0$ satisfying that the Ricci curvature is bounded below by $-(d-1)L$. Let Δ be the Laplace-Beltrami operator and $\Box$ the Hodge-Kodaira Laplacian acting on differential 1-forms. For $p \in [1, \infty]$, let L^p, $\mathscr{A}^{0,p}$ and $\mathscr{X}^{0,p}$ be the Lebesgue space of scalar functions, that of 1-forms and that of vector fields respectively(with respect to m, otherwise stated). For $r > 0$, we define (r, p)-Sobolev spaces by

- $W^{r,p} := (1-\Delta)^{-r/2} L^p$,
- $\mathscr{A}^{r,p} := (L+1-\Box)^{-r/2} \mathscr{A}^{0,p}$.

For $f \in W^{r,p}$, $\|f\|_{r,p} := \|(1-\Delta)^{r/2} f\|_{L^p}$. In the same way, we define the Sobolev norm $\|\cdot\|_{r,p}$ (use the same symbol) on $\mathscr{A}^{r,p}$. For $p \in (1, \infty)$, let $\mathscr{X}^{-r,p} = \left(\mathscr{A}^{r,p'}\right)^*$ be the Sobolev space of vector fields of negative differentiability orders for $1/p + 1/p' = 1$. Let us introduce the following condition on a pair of indices (r, p):

(I) $p \in (1, 2)$ and $r > 2 + d - d/p$.

Let $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$ be the Brownian motion, or the Markov process generated by $\Delta/2$. For $\alpha \in \mathscr{A}$, we can define the stochastic line integral $\int_{X[0,t]} \alpha$ along $\{X_s\}_{s \in [0,t]}$. We regard the stochastic line integral as a Random map $R_t : \mathscr{A} \to \mathbb{R}$ given by $R_t(\alpha) = \int_{X[0,t]} \alpha$. Let $\mathscr{C}^{-r,p} := C([0, \infty) \to \mathscr{X}^{-r,p})$ with the compact uniform convergence topology.

Theorem 1 [1] *For each pair (r, p) satisfying (I), R is realized as a $\mathscr{C}^{-r,p}$-valued random variable by taking a suitable version.*

We denote the space of the probability measures by $\mathscr{M}_+^1$ and the space of the signed measures on M of finite total variation by $\mathscr{M}$. We consider the weak topology on $\mathscr{M}_+^1$ and $\mathscr{M}$. Let $R_t^\lambda := \lambda^{-1} R_{\lambda t}$ and $L_t^\lambda := \lambda^{-1} \int_0^{\lambda t} \delta_{X_s} ds$. Then $(R_1^\lambda, L_1^\lambda)$ is an $\mathscr{X}^{-r,p} \times \mathscr{M}_+^1$-valued random variable.

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Definition 1 *For* $(\zeta, \mu) \in \mathscr{X}^{-r,p} \times \mathscr{M}$*, we say* $(\zeta, \mu) \in \mathscr{G}$ *if and only if it satisfies the following:*

(i) $\mu \in \mathscr{M}_+^1$, $\mu \ll m$ *and* $h_\mu := \sqrt{d\mu/dm} \in W^{1,2}$.

(ii) $\operatorname{div} \zeta = 0$ *in the sense of distribution, i.e.,* ${}_{\mathscr{X}^{-r,p}}\langle \zeta, du \rangle_{\mathscr{A}^{r,p'}} = 0$ *for each* $u \in C^\infty(M)$.

(iii) *Let* $\bar{\zeta}$ *be given by*

$$\bar{\zeta}(x) = \begin{cases} \dfrac{1}{h_\mu(x)^2}\zeta(x) & \text{if } h_\mu(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{\zeta} \in \mathscr{X}^{0,2}(d\mu)$.

We define a rate function I *on* $\mathscr{X}^{-r,p} \times \mathscr{M}$ *by*

$$I(\zeta, \mu) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\{h_\mu > 0\}} \frac{|\zeta|^2}{h_\mu^2} dm + \frac{1}{2}\int_M |\nabla h_\mu|^2 \, dm & \text{if } (\zeta, \mu) \in \mathscr{G}, \\ \infty & \text{otherwise.} \end{cases}$$

Note that ζ *is regarded as a measurable vector field.*

Theorem 2 [1] *For each pair* (r, p) *satisfying (I),* $\{(R_1^\lambda, L_1^\lambda)\}_{\lambda > 0}$ *satisfies the large deviation principle in* $\mathscr{X}^{-r,p} \times \mathscr{M}_+^1$ *as* $\lambda \to \infty$ *with the convex good rate function* I*. That is, for each* $E \in \mathscr{X}^{-r,p} \times \mathscr{M}_+^1$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \left(\sup_{x \in M} \mathbb{P}_x \left[(R_1^\lambda, L_1^\lambda) \in E \right] \right) \le - \inf_{(\zeta, \mu) \in \bar{E}} I(\zeta, \mu),$$
$$\liminf_{t \to \infty} \frac{1}{t} \log \left(\inf_{x \in M} \mathbb{P}_x \left[(R_1^\lambda, L_1^\lambda) \in E \right] \right) \ge - \inf_{(\zeta, \mu) \in E^\circ} I(\zeta, \mu),$$

where $\bar{E}$ *is the closure of* E *and* E° *the interior of* E.

In [2], the large deviation principle in the case of $p = 2$ is studied in a bit different formulation. We use this result for proving Theorem 2 via (inverse) contraction principle. The key estimate for both of Theorem 1 and Theorem 2 is the following exponential integrability:

Proposition 1 *Let* $G_r(\zeta) := \sup_{\|\alpha\|_{r,\infty} \le 1} |\zeta(\alpha)|$*. Then, for* $r > 2$*, there exist constants* $\gamma > 0$ *and* $C > 0$ *so that for each* $\eta \in (0, 1]$,

$$\sup_{x \in M} \mathbb{E}_x \left[\exp \left(\gamma \eta^{-1/2} \sup_{0 \le t \le \eta} G_r(R_t) \right) \right] < C.$$

References

[1] S. Kusuoka, K. Kuwada, and Y. Tamura, *Large deviation principle for currents generated by stochastic line integrals of 1-forms on compact riemannian manifolds*, preprint.

[2] K. Kuwada, *On large deviations for random currents induced from stochastic line integrals*, Forum Mathematicum **18** (2006), no. 4, 639–676.