Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths

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Let (X, H, μ) be an abstract Wiener space, Y be a real separable Banach space, $w = (w_t)_{0\le t\le 1}$ be an X-valued Brownian motion. Let also $X^\varepsilon := (X^\varepsilon_t)_{0\le t\le 1}$ $(\varepsilon > 0)$ be a solution of the following formal (Stratonovich) SDE:

$$dX^\varepsilon_t = \sigma(X^\varepsilon_t) \circ \varepsilon dw_t + \sum_{i=1}^{N} a_i(\varepsilon) b_i(X^\varepsilon_t) dt \quad \text{with } X^\varepsilon_0 = 0. \tag{1}$$

Here, $\sigma \in C^\infty_b(Y, L(X, Y)), b_i \in C^\infty_b(Y, Y), i = 1, \dots, N$, and $a = (a_1, \dots, a_N) : [0, 1] \to \mathbb{R}^N$ is a smooth curve. In this talk, we will discuss the Laplace type asymptotic expansion of the functional integral of the form $\mathbb{E}\big[G(X^\varepsilon) \exp(-F(X^\varepsilon/\varepsilon^2))\big]$ as $\varepsilon \searrow 0$. The large deviation was done in Inahama-Kawabi [J. London Math. Soc., 2006]. The Laplace method for the leading term $(= \alpha_0)$ was done in Inahama [JFA, 2006]、Inahama-Kawabi [Proceedings of the Abel Symposium 2005, to appear].

In order to give a precise definition for the Wiener functional X^ε, we introduce some notations. For a real separable Banach B, we set $P(B) := \{x \in C([0, 1], B) \mid x_0 = 0\}$, $BV(B) := \{\gamma \in P(B) \mid \|\gamma\|_1 < \infty\}$, and $G\Omega_p(B)$ $(2 < p < 3)$, (which is called the space of geometric rough paths over B). The law of εw on $P(X)$ is denoted by $\mathbb{P}'_\varepsilon$. $\mathcal{H}$ is the Cameron-Martin space for $(P(X), \mathcal{H}, \mathbb{P}'_1)$. In this talk we assume $(|\cdot|_{X\otimes X}, \mu)$ satisfies the exactness condition **(EX)** (cf. Ledoux-Lyons-Qian), which implies the existence of Brownian rough paths $\overline{w} = (1, \overline{w}_1, \overline{w}_2) \in G\Omega_p(X)$. The law of scaled Brownian rough paths $\overline{\varepsilon w}$ on $G\Omega_p(X)$ is denotedd by $\mathbb{P}_\varepsilon$.

Set $\tilde{X} := X \oplus \mathbb{R}^N$ and define $\tilde{\sigma} \in C^\infty_b(Y, L(\tilde{X}, Y))$ by

$$\tilde{\sigma}(y)\big[(x, u)\big]_{\tilde{X}} := \sigma(y)x + \sum_{i=1}^{N} b_i(y)u_i, \quad y \in Y, x \in X, u = (u_1, \dots, u_N) \in \mathbb{R}^N.$$

We consider the following ODE in the rough path sense:

$$dy_t = \tilde{\sigma}(y_t) d\tilde{x}_t \quad \text{with } y_0 = 0. \tag{2}$$

For an input $\overline{\tilde{x}} \in G\Omega_p(\tilde{X})$, there is a unique solution $\overline{z} = (\overline{\tilde{x}}, \overline{y}) \in G\Omega_p(\tilde{X} \oplus Y)$ in the rough path sense. We denote it by $\overline{y} = \Phi(\overline{\tilde{x}})$ and also call it a solution of ODE (2). The Itô map $\Phi : G\Omega_p(\tilde{X}) \to G\Omega_p(Y)$ is locally Lipschitz continuous. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(N)}) \in \mathrm{BV}(\mathbb{R}^N), \overline{x} \in G\Omega_p(X)$, define $\iota(\overline{x}, \lambda) \in G\Omega_p(\tilde{X})$ by $\iota(\overline{x}, \lambda)_1(s, t) = (\overline{x}_1(s, t), \lambda_t - \lambda_s)$,

$$\iota(\overline{x}, \lambda)_2(s, t) = \Big(\overline{x}_2(s, t), \int_s^t \overline{x}_1(s, u) \otimes d\lambda_u, \int_s^t (\lambda_u - \lambda_s) \otimes \overline{x}_1(s, du), \int_s^t (\lambda_u - \lambda_s) \otimes d\lambda_u\Big).$$

This map $\iota : G\Omega_p(X) \times \mathrm{BV}(\mathbb{R}^N) \to G\Omega_p(\tilde{X})$ is also locally Lipschitz continuous.

For $\varepsilon \ge 0$, we set $\lambda^\varepsilon(t) := (a_1(\varepsilon)t, \dots, a_N(\varepsilon)t)$ and set $\Psi_\varepsilon : \mathrm{BV}(X) \to \mathrm{BV}(Y)$ by

$$\Psi_\varepsilon(h)_t := \Phi\big(\iota(\overline{h}, \lambda^\varepsilon)\big)_1(0, t), \qquad 0 \le t \le 1.$$

Then, $y := \Psi_\varepsilon(h)$ satisfies the following Y-valued usual ODE:

$$dy_t = \sigma(y_t) dh_t + \sum_{i=1}^{N} a_i(\varepsilon) b_i(y_t) dt \quad \text{with } y_0 = 0.$$

With these observations in mind, we define our Wiener functional X^ε as follows:

$$X^\varepsilon_t := \Phi\big(\iota(\overline{\varepsilon w}, \lambda^\varepsilon)\big)_1(0, t), \qquad 0 \le t \le 1.$$

Examples of X^ε include heat processes over loop spaces and solutions usual SDEs (1) over finite dimensional spaces or M-type 2 Banach spaces. (Because of Wong-Zakai's approximation theorem.)

In this talk we assume the following: Below, the Fréchet derivative of $BV(X), P(Y)$ is denoted by D、and that of Y is by denoted ∇.

(H1): F and G are real-valued bounded continuous functions defined on $P(Y)$.

(H2): The function $F_\Lambda := F \circ \Psi_0 + \|\cdot\|^2_{\mathcal{H}}/2$ defined on $\mathcal{H}$ attains its minimum at a unique point $\gamma \in \mathcal{H}$. For this γ, we write $\phi := \Psi_0(\gamma)$.

(H3): The functions F and G are $n+3$ and $n+1$ times Fréchet differentiable on a neighborhood $B(\phi)$ of $\phi \in P(Y)$, respectively. Moreover there exist positive constants $M_1, \ldots, M_{n+3}$ such that

$$
\begin{aligned}
\big|D^k F(\eta)[y, \ldots, y]\big| &\le M_k \|y\|^k_{P(Y)}, \quad k = 1, \ldots, n+3, \\
\big|D^k G(\eta)[y, \ldots, y]\big| &\le M_k \|y\|^k_{P(Y)}, \quad k = 1, \ldots, n+1,
\end{aligned}
$$

hold for any $\eta \in B(\phi)$ and $y \in P(Y)$.

(H4): At the point $\gamma \in \mathcal{H}$, we consider the Hessian $A := D^2(F \circ \Psi_0)(\gamma)|_{\mathcal{H}\times\mathcal{H}}$. As a bounded self-adjoint operator on $\mathcal{H}$, the operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}}$ in the form sense. (Actually, A is Hilbert-Schmidt. By the min-max principle, it is equivalent to assume that infimum of all eigenvalues of A are strictly larger than -1.)

Our main result is as follows (Inahama-Kawabi [JFA, to appear]). We do not give the explicit forms of the constants α_n, $c(\gamma)$ here, because they are too long.

Theorem 1 *Under the assumptions* **(EX)**, **(H1)**–**(H4)**, *we have the following asymptotic expansion:*

$$
\begin{aligned}
&\mathbb{E}\Big[G(X^\varepsilon)\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\Big] \\
&= \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big)\exp\big(-c(\gamma)/\varepsilon\big)\cdot\big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_n\varepsilon^n + O(\varepsilon^{n+1})\big). \qquad (3)
\end{aligned}
$$

The key of the proof is the (stochastic) Taylor expansion with respect to the topology of the rough path space given as below. (cf. Aida[preprints, 2005 & 2006]、Inahama-Kawabi [JFA, to appear])

$$
\Phi(\iota(\overline{\gamma + \varepsilon w}, \lambda^\varepsilon))_1 = \phi + \varepsilon\phi^1(\overline{w})_1 + \varepsilon^2\phi^2(\overline{w})_1 + \cdots + \varepsilon^n\phi^n(\overline{w})_1 + O(\varepsilon^{n+1}).
$$

Since this expansion is deterministic, it is natural to guess that the same mothod applies to asymptotic problems of other probability measures on $G\Omega_p(X)$.

Our method of the stochastic Taylor expansion is slightly different from Aida's method. He uses the derivative equation, whose coefficient is of course of linear growth. Since it is not known whether Lyons' continuity theorem holds or not for unbounded coefficients, he extends the continuity theorem for the case of the derivative equation in the first preprint. On the other hand, we use the method in Azencott's original paper [1982, LNM921] and we only need the continuity theorem for the given equation, whose coefficient is bounded. The price we have to pay is that notations and proofs may seem slightly long. However, the strategy of this method is quite simple and straight forward.