Generalized positive continuous additive functionals of multidimensional Brownian motion and their associated Revuz measure

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(1) generalized PCAF. Let $(W_0^N, \mathcal{F}_t, P)$ be the N dimensional standard Wiener space, i.e., $W_0^N = \{W_t = (W_t^1, W_t^2, \ldots, W_t^N) : [0,\infty) \to \mathbb{R}^N | W_t$ is continuousand $W_0 = 0\}$, $\mathcal{F}_t = \sigma\{W_s; 0 \le s \le t\}$ and P is the standard Wiener measure. Let $\boldsymbol{D}_2^\gamma$ be the Meyer-Watanabe's Sobolev space. Let $\mathbb{R}_A^N$ be a subset of $\mathbb{R}^N$ satisfying $|\mathbb{R}^N \setminus \mathbb{R}_A^N| = 0$. We consider $A = \{A(t,x;W.); t \ge 0, x \in \mathbb{R}_A^N\} \subset \boldsymbol{D}_2^\gamma$.

Definition 1. (i) Let $\gamma < 0$. A is called a $\boldsymbol{D}_2^\gamma$ additive functional of the N dimensional Brownian motion if and only if (a) $A(t,x) = A(t,x;W.)$ is $\mathcal{F}_t$ measurable, (b) $A(0,x) = 0$ and

$$(c) \qquad A(t+s, x; W.) - A(t,x;W.) = A(s, x+W_t; (\theta_t W).)$$

in $\boldsymbol{D}_2^\gamma$, where $(\theta_t W)_s = W_{t+s} - W_t$.
(ii) $A(t,x)$ is positive if $\langle F, A(t,x)\rangle \ge 0$ for all $F(\in \boldsymbol{D}_2^{-\gamma}) \ge 0$.

Remark 1. $A(s, x+W_t; (\theta_t W).)$ is defined by $\int A(s, x+y; (\theta_t W).) \diamond_1 \delta(W_t - y)dy$, where $\diamond_1$ denotes the Wiener product (see [1]). If we assume Condition 1 below, then $A(s, x + W_t; (\theta_t W).) = \sum I_{\boldsymbol{n}}^{\theta_t W}(a_{\boldsymbol{n}}(t, x+W_t))$, where $A(t,x;W.) = \sum I_{\boldsymbol{n}}(a_{\boldsymbol{n}}(t,x))$.

Condition 1. $\int \|A(t,y)\|_{2,\gamma}^2 \mathrm{e}^{-\delta|y|^2} dy < \infty$ for all $\delta > 0$, where $\|\cdot\|_{2,\gamma}$ denotes the norm of $\boldsymbol{D}_2^\gamma$.

Proposition 1. *We assume $A(t,x)$ is continuous w.r.t. t in $\boldsymbol{D}_2^\gamma$. Let $\Delta = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t\}$ be a partition of $[0,t]$, and put $|\Delta| = \max |t_{i+1} - t_i|$. Then we obtain*

$$\lim_{|\Delta| \to 0} \sum_{i=0}^n a_0(t_{i+1} - t_i, x + W_{t_i}) = A(t,x)$$

in $\boldsymbol{D}_2^\gamma$.

(2) Revuz measure associated to generalized PCAF. Let A be a $\boldsymbol{D}_2^\gamma$ positive continuous additive functional (abbreviated $\boldsymbol{D}_2^\gamma$ PCAF). For all $f \in \mathcal{D}$

$$\int_0^t \langle f(x+W_s), dA_s(x)\rangle = \lim_{|\Delta|\to 0} \sum_{i=0}^n \langle f(x+W_{t_i}), A(t_{i+1},x) - A(t_i,x)\rangle$$

is well defined. Moreover, under Condition 1,

$$f(\in \mathcal{D}) \mapsto \int_{\mathbb{R}^N} \int_0^t \langle f(x+W_s), dA_s(x)\rangle dx \in \mathcal{D}'.$$

Definition 2. Assume Condition 1. The Revuz measure ν_A associated to $\boldsymbol{D}_2^\gamma$ PCAF A is the measure on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} f(x)\nu_A(dx) = \int_{\mathbb{R}^N}\int_0^1 \langle f(x+W_s), dA_s(x)\rangle dx$$

for all $f \in \mathcal{D}$.

Proposition 2. *Under Condition 1*

$$\int_{\mathbb{R}^N} f(x)\nu_A(dx) = \lim_{|\Delta|\to 0}\sum_{i=0}^{n}\int_{\mathbb{R}^N} f(x)a_0(t_{i+1}-t_i, x)dx,$$

Δ *denoting a partition of* $[0,1]$.

(3) Local time representation of generalized PCAF.

Condition 2. For all $\eta > 0$ small enough and for all $\delta > 0$,

$$\int |y-x|^{2-N-\eta}\mathrm{e}^{-\delta|y-x|^2}\mu(dy) < \infty.$$

Theorem 1. *Let $A = \{A(t,x;W.); t \geq 0, x \in \mathbb{R}_A^N\}$ be a $\boldsymbol{D}_2^\gamma$ PCAF satisfying Condition 1. Assume ν_A satisfies Condition 2. Then it holds that $\int L(t,y-x)\nu_A(dy)$ exists in $\boldsymbol{D}_2^\alpha$ ($\alpha < 1-N/2$) and that*

$$A(t,x) = \int L(t,y-x)\nu_A(dy),$$

where $L(t,z)$ denotes the local time of W at z.

(4) generalized PCAF corresponding to Radon measure. Let $T \in \mathcal{D}'$ be a positive distribution and μ_T be the corresponding Radon measure. Let $\alpha < 1-N/2$. Then we obtained a $\boldsymbol{D}_2^\alpha$ PCAF $A_T(t,x)$ corresponding to T under Condition 2 ([2]). Applying Mehler's formula, we have the followings:

Theorem 2. *Assume μ_T satisfies Condition 2. Then $A_T(t,x) \in \boldsymbol{D}_2^{-\beta}$ if and only if*

$$\int_0^\infty \iint \int_0^t \int_0^s \mathrm{e}^{-r} r^{\beta-1} p_N(s-\mathrm{e}^{-2r}u, y-\mathrm{e}^{-r}z-(1-\mathrm{e}^{-r})x) \times p_N(u, z-x)duds\mu_T(dz)\mu_T(dy)dr < \infty.$$

Theorem 3. *Assume μ_T satisfies Condition 2. If*

$$\iint |y-z|^{2-N-\eta}|z-x|^{2-N-\eta}\mathrm{e}^{-\delta|y-z|^2}\mathrm{e}^{-\delta|z-x|^2}\mu_T(dy)\mu_T(dz) < \infty,$$

then $A_T(t,x) \in L^2(P)$.

References

[1] H. Uemura, *Unrenormalized intersection local time of Brownian motion and its local time representation*, J. Math. Kyoto Univ., vol. 43-4, (2003), 671-687.

[2] H. Uemura, *Positive continuous additive functionals of multidimensional Brownian motion and the Brownian local time*, preprint.