Witten Laplacians on pinned path groups

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Let G be a simply connected and connected compact Lie group. Let $a \in G$. We denote the unit element of G by e. We consider a pinned path group

$$P_{e,a}(G) = C([0,1] \to G \mid \gamma(0) = e, \gamma(1) = a). \qquad (1)$$

Let $\lambda > 0$ and $\nu_{\lambda,a}$ be the pinned Brownian motion measure on $P_{e,a}(G)$ such that

$$\begin{aligned}
&\nu_{\lambda,a}\left(\{\gamma \in P_{e,a}(G) \mid \gamma(t_1) \in A_1, \ldots, \gamma(t_{n-1}) \in A_{n-1}\}\right) \\
&\quad = p\left(\lambda^{-1}, e, a\right)^{-1} \int_{G^n} \prod_{i=1}^{n-1} p\left(\lambda^{-1}\left(t_i - t_{i-1}\right), x_{i-1}, x_i\right) 1_{A_i}(x_i) \cdot p\left(\lambda^{-1}\left(1 - t_{n-1}\right), x_{n-1}, a\right) \\
&\qquad\qquad dx_1 \cdots dx_{n-1},
\end{aligned}$$

where $0 = t_0 < t_1 < \cdots < t_{n-1} < 1$, $x_0 = e$ and $A_i \subset G$. $p(t,x,y)$ denotes the heat kernel of $e^{(t/2)\Delta}$.

We are interested in the following Witten Laplacian.

Definition 1 *Let d be a exterior derivative on $P_{e,a}(G)$. Let $d^*_{\nu_{\lambda,a}}$ denote the adjoint operator of d on $L^2(\wedge T^* P_{e,a}(G), d\nu_{\lambda,a})$. Set $\Box_{\lambda,a} = -\left(dd^*_{\nu_{\lambda,a}} + d^*_{\nu_{\lambda,a}} d\right)$. For an open set Ω in $P_{e,a}(G)$, set*

$$E_{Dir,1}(\lambda, \Omega) \;=\; \inf\left\{ (-\Box_{\lambda,a}\alpha, \alpha)_{L^2(\nu_{\lambda,a})} \;\middle|\; \alpha \text{ is a smooth 1-form and } \alpha|_{\partial\Omega} = 0 \right\}.$$

Definition 2 *For $\gamma, \eta \in P_{e,a}(G)$, define $d(\gamma, \eta) = \max_{0 \le t \le 1} d(\gamma(t), \eta(t))$. Also let $E(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt$ and*

$$\Omega_L = \left\{\gamma \in P_{e,a}(G) \mid \sqrt{E(\gamma)} \le L\right\} \qquad (2)$$

$$B_\varepsilon(\eta) = \{\gamma \in P_{e,a}(G) \mid d(\gamma, \eta) < \varepsilon\} \qquad (3)$$

$$\Omega_{L,\varepsilon} = \left\{\gamma \in P_{e,a}(G) \mid \text{there exists } \eta \in \Omega_L \text{such that } \gamma \in B_\varepsilon(\eta)\right\}. \qquad (4)$$

Also we define for $0 < \alpha < 1$,

$$\|\gamma\|_\alpha = \sup_{0 \le s,t \le 1} \frac{d(\gamma(t), \gamma(s))}{|t - s|^\alpha}. \qquad (5)$$

Remark 3 *Let $R > 0$. Set*

$$D_{\alpha,R} \quad = \quad \{\gamma \in P_{e,a}(G) \mid \|\gamma\|_\alpha < R\}.$$

Then for any small $\varepsilon > 0$, there exists $L > 0$ such that $D_{\alpha,R} \subset \Omega_{L,\varepsilon}$.

The following is one of our main result.

Theorem 4 *Suppose that a is outside the cut-locus of e. We denote the all geodesics connecting e and a by $\{c_i\}_{i=1}^\infty \subset P_{e,a}(G)$. Also we denote the all eigenvalues of $(\nabla^2 E)(c_i)$ by $\{\xi_j(c_i) \mid j = 1, 2, \ldots\}$. Let Ω be an open subset of $P_{e,a}(G)$. Assume that there exists a sufficeintly small $\varepsilon > 0$ and positive L such that $\Omega \subset \Omega_{L,\varepsilon}$ and $\partial\Omega$ does not contain any geodesics. Then it holds that*

$$\lim_{\lambda\to\infty} \frac{E_{Dir,1}(\lambda, \Omega)}{\lambda} = \min \{\theta_1(c_i) \mid c_i \in \Omega\},$$

where

$$\theta_1(c_i) = \inf_{j\ge 1} \left(\max(\xi_j(c_i), 0) + \sum_{\{k\ne j, \xi_k(c_i)<0\}} |\xi_k(c_i)| \right).$$

Further, it holds that $\theta_1(c_i) > 0$ for all $i \ge 1$.

Remark 5 *When $G = SU(n, \mathbb{C}), SO(n)$, it holds that*

$$\theta_1(c_i) \ge \frac{l(c_i)}{2\pi \dim G} - C_1 \to +\infty \quad (i \to \infty). \tag{6}$$

where $l(c_i)$ denotes the length of the geodesic c_i and C is a constant.

The main theorem can be proved by applying the following log-Sobolev inequality (with a potential function). Below, a is not necessarily outside the cut-locus of e.

Theorem 6 *There exist constants $C_1, C_2 > 0$ such that for any sufficiently large $\lambda > 0$ and $f \in \mathfrak{F}C_b^\infty(P_{e,a}(G))$, it holds that*

$$\int_{P_{e,a}(G)} f^2(\gamma) \log \left(\frac{f^2(\gamma)}{\|f\|^2_{L^2(\nu_{\lambda,a})}} \right) d\nu_{\lambda,a}(\gamma) \quad \le \quad \frac{2}{\lambda} \left(1 + \frac{C_1}{\lambda}\right) \mathcal{E}_{\lambda,V_{\lambda,a}}(f, f), \tag{7}$$

where

$$\mathcal{E}_{\lambda,V_{\lambda,a}}(f, f) \quad = \quad \int_{P_{e,a}(G)} |(\nabla f)(\gamma)|^2_H d\nu_{\lambda,a} + \int_{P_{e,a}(G)} \lambda^2 V_{\lambda,a}(\gamma) f(\gamma)^2 d\nu_{\lambda,a}, \tag{8}$$

$$V_{\lambda,a}(\gamma) = \frac{1}{4} \left\{ |b(1)|^2 + \frac{2}{\lambda} \log\left(\lambda^{-d/2} p(1/\lambda, e, a)\right) \right\} + \frac{C_2}{\lambda} \left\{ 1 + |b(1)|^2 + \left(\int_0^1 |b(s)| ds \right)^2 \right\}.$$