INVARIANT SETS AND ERGODIC DECOMPOSITION OF LOCAL SEMI-DIRICHLET FORMS

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1. Weakly invariant set and strongly invariant set

Let X be a separable metric space and m a σ-finite Borel measure on X. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on $L^2(X;m)$ and $(\hat{T}_t)_{t\geq 0}$ the dual C_0-semigroup of $(T_t)_{t\geq 0}$ on $L^2(X;m)$. An m-measurable subset B of X is said to be *weakly invariant* with respect to $(T_t)_{t\geq 0}$ if $I_{B^c}T_tI_Bu=0$ for any $t>0$ and $u\in L^2(X;m)$, equivalently B^c is weakly invariant with respect to $(\hat{T}_t)_{t\geq 0}$. An m-measurable subset B of X is said to be *(strongly) invariant* with respect to $(T_t)_{t\geq 0}$ if $T_tI_Bu=I_BT_tu$ for any $t>0$ and $u\in L^2(X;m)$. Clearly, the strong invariance implies the weak one and B is strongly invariant if and only if both B and B^c are weakly invariant. So if $(T_t)_{t\geq 0}$ is symmetric, then the weak invariance is equivalent to the strong one. Fix $\gamma\geq 0$. A bilinear form $(\mathcal{E},\mathcal{F})$ is said to be a *positivity preserving form with a lower bound $-\gamma$ on $L^2(X;m)$* if $(\mathcal{E}_\gamma,\mathcal{F})$ is a coercive closed form having the property that for $u\in\mathcal{F}$, $u^+,u^-\in\mathcal{F}$ and $\mathcal{E}(u^+,u^-)\leq 0$, equivalently $\mathcal{E}(u,u^+)\geq -\gamma\|u^+\|_2^2$. A bilinear form $(\mathcal{E},\mathcal{F})$ is said to be a *semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$* if $(\mathcal{E}_\gamma,\mathcal{F})$ is a coercive closed form having the property that for $u\in\mathcal{F}$, $u^+\wedge 1\in\mathcal{F}$ and $\mathcal{E}(u^+\wedge 1,u-u^+\wedge 1)\geq 0$, equivalently, $\mathcal{E}(u+u^+\wedge 1,u-u^+\wedge 1)\geq -\gamma\|u-u^+\wedge 1\|_2^2$. Any semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$ is automatically a positivitiy preserving form with a lower bound $-\gamma$ on $L^2(X;m)$.

Let $(\mathcal{E},\mathcal{F})$ be a semi-Dirichlet form or positivity preserving form with a lower bound $-\gamma$ on $L^2(X;m)$. Then there exists a C_0-semigroup $(T_t)_{t\geq 0}$ on $L^2(X;m)$ such that $(e^{-\gamma t}T_t)_{t\geq 0}$ is contractive on $L^2(X;m)$ and its resolvent $(G_\alpha)_{\alpha>\gamma}$ defined by $G_\alpha f:=\int_0^\infty e^{-\alpha t}T_tf\,dt$, $f\in L^2(X;m)$ satisfies $G_\alpha f\in\mathcal{F}$ and $\mathcal{E}_\alpha(G_\alpha f,g)=(f,g)$ for $f\in L^2(X;m)$, $g\in\mathcal{F}$, $\alpha>\gamma$. If $(\mathcal{E},\mathcal{F})$ is a positivity preserving (resp. semi-Dirichlet) form with a lower bound $-\gamma$ on $L^2(X;m)$, then $(T_t)_{t\geq 0}$ satisfies that $T_tu\geq 0$ if $u\geq 0$ (resp. $0\leq T_tu\leq 1$ if $0\leq u\leq 1$) for any $t>0$ and $u\in L^2(X;m)$. For the details, see [1],[5],[6].

For a positivity preserving (resp. semi-Dirichlet) form with a lower bound $-\gamma$ on $L^2(X;m)$ is said to be *quasi-regular* if $(\mathcal{E}_\gamma,\mathcal{F})$ is a quasi-regular positivity preserving (resp. semi-Dirichlet) form on $L^2(X;m)$ in the sense of Ma-Röckner [10] (resp. Ma-Overbeck-Röckner [8]). For a positivity preserving form with a lower bound $-\gamma$ on $L^2(X;m)$ is said to be *local* if $(\mathcal{E}_\gamma,\mathcal{F})$ is local in the sense of Chapter V. Definition 1.1 in [9].

Theorem 1.1. *Suppose that $(T_t)_{t\geq 0}$ is associated with a quasi-regular local positivity preserving form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$. Then the weak invariance with respect to $(T_t)_{t\geq 0}$ is equivalent to the strong invariance.*

2. ERGODIC DECOMPOSITION

As an application of Theorem 1.1, we give an ergodic decomposition of the right process associated with a quasi-regular semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$. Take a point $\Delta \notin X$ which is added to X as an isolated point. If X is a locally compact, then it is added to X as one point compactification. We consider a Borel right m-special standard process $\mathbf{M} = (\Omega, X_t, P_x)_{x\in X_\Delta}$. $\mathbf{M}$ is said to be *associated with* a semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ if $T_t u = E_x[u(X_t)]$ m-a.e. $x \in X$ for Borel function $u \in L^2(X;m)$. We say that $\mathbf{M}$ satisfies the absolute continuity condition with respect to m if $p_t(x,dy) \ll m(dy)$ for any $x \in X$ and $t > 0$. A set $B \subset X$ is said to be $\mathbf{M}$-*invariant* if B is nearly Borel and

$$P_x(X_t \in B \text{ for } {}^\forall t \in [0,\zeta[,\ X_{t-} \in B \text{ for } {}^\forall t \in]0,\zeta[) = 1,\ x \in B.$$

Obviously any complement of an $\mathbf{M}$-invariant set is weakly invariant relative to $(T_t)_{t\geq 0}$ associated with $(\mathcal{E},\mathcal{F})$ or $\mathbf{M}$. A set $N \subset X$ is called *(m-)properly exceptional* if N is a nearly Borel m-negligible set and $X \setminus N$ is $\mathbf{M}$-invariant. If $\mathbf{M}$ has a decomposition $X = B_1 + B_2 + N$ such that each B_i is $\mathbf{M}$-invariant and N is properly exceptional, then B_i is strongly invariant relative to $(T_t)_{t\geq 0}$.

From now on, let $(\mathcal{E},\mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$ and assume that there exists a Borel right m-special standard process $\mathbf{M}$ associated with $(\mathcal{E},\mathcal{F})$.

Theorem 2.1 (Ergodic decomposition I)**.** $\mathbf{M}$ *admits the following decomposition: there exist* $\mathbf{M}$*-invariant sets* X_c^*, X_d^* *and a properly exceptional set* N *such that*

(1) $X = X_c^* + X_d^* + N$.
(2) $\mathbf{M}_{X_d^*}$ *is transient in the following sense: there exists a bounded Borel function* $u \in L^1(X;m)$ *such that* $u > 0$ $\mathcal{E}$*-q.e. on* X_d^*, $u = 0$ $\mathcal{E}$*-q.e. on* X_c^* *and* $0 < Ru < \infty$ $\mathcal{E}$*-q.e. on* X_d^*. *More strongly, we can take such* u *with* $Ru \in L^\infty(X_d^*;m)$.
(3) $\mathbf{M}_{X_c^*}$ *is recurrent in the following sense: for any* m*-a.e. nonnegative Borel function* u, $Ru = 0$ *or* ∞ $\mathcal{E}$*-q.e. on* X_c^*. *If further* u *is* m*-a.e. strictly positive on* X_c^*, *then* $Ru = \infty$ $\mathcal{E}$*-q.e. on* X_c^*.

Further assume that $\mathbf{M}$ *satisfies the absolute continuity condition with respect to* m. *Then* X_c^* *and* X_d^* *can be taken to be finely open and finely closed and* N *can be taken to be empty. In this case,* $\mathbf{M}_{X_d^*}$ *is transient in the sense of Getoor* [4], *that is, there exists a nonnegative bounded Borel function* u *on* X_d^* *such that* Ru *is strictly positive and bounded on* X_d^*. *Also* $\mathbf{M}_{X_c^*}$ *is recurrent in the following sense: for any* m*-a.e. nonnegative Borel function* u *on* X_c^*, $Ru(x) = 0$ *or* ∞ *for* $x \in X_c^*$. *If further* u *is* m*-a.e. strictly positive on* X_c^*, *then* $Ru(x) = \infty$ *for* $x \in X_c^*$. *Finally we note that* X_c^* *and* X_d^* *can be taken to be open and closed if* $\mathbf{M}$ *has the strong Feller property.*

If $(T_t)_{t\geq 0}$ is also a family of contractive operators on $L^1(X;m)$, then we have the following assertion without assuming the local property of $(\mathcal{E},\mathcal{F})$:

Theorem 2.2 (Ergodic decomposition II)**.** *Suppose that* $(T_t)_{t\geq 0}$ *forms a family of contractions on* $L^1(X;m)$, *equivalently,* $(\mathcal{E},\mathcal{F})$ *is a (non-symmetric) Dirichlet form with a lower bound* $-\gamma$ *on* $L^2(X;m)$. *Then the same conclusion as in Theorem 2.1* *holds. More strongly, we have that for any m-a.e. nonnegative $g \in L^1(X;m)$, $Rg < \infty$ $\mathcal{E}$-q.e. on X_d^*.*

Any coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is said to be *strictly irreducible* (resp. *irreducible*) if for any weakly (resp. strongly) invariant set B relative to the C_0-semigroup $(T_t)_{t\geq 0}$ of $(\mathcal{E},\mathcal{F})$, $m(B)=0$ or $m(B^c)=0$.

Theorem 2.3 (Transience of part processes)**.** *Assume one of the following:*

- $(\mathcal{E},\mathcal{F})$ *is local and irreducible.*
- $(\mathcal{E},\mathcal{F})$ *is a strictly irreducible (non-symmetric) Dirichlet form.*

Take an open set G such that $X\setminus G$ is non-$\mathcal{E}$-polar. Then the conservative part G_c^* in the ergodic decomposition $G=G_c^*+G_d^*+N$ for the part process $\mathbf{M}_G$ is $\mathcal{E}$-polar. In particular, $\mathbf{M}_G$ is transient in the sense specified in Theorem 2.1(2). Further assume that $\mathbf{M}$ satisfies the absolute continuity condition with respect to m. Then $G=G_d^*$, and $\mathbf{M}_G$ is transient in the sense of Getoor [4].

Theorem 2.4 (Comparison of transience)**.** *Let m be a σ-finite Borel measure on X and $(\mathcal{E}^{(i)},\mathcal{F}^{(i)})$ be a quasi-regular semi-Dirichlet form with the same lower bound* 0 *on $L^2(X;m)$ and assume that there exists a Borel right m-special standard process $\mathbf{M}^{(i)}$ associated with $(\mathcal{E}^{(i)},\mathcal{F}^{(i)})$, for each $i=1,2$. Assume that $(\mathcal{E}^{(1)},\mathcal{F}^{(1)})$ satisfies the strong sector condition. Suppose that $\mathcal{F}^{(2)}\subset\mathcal{F}^{(1)}$ and $(\mathcal{E}^{(1)})^{1/2}\leq(\mathcal{E}^{(2)})^{1/2}$ on $\mathcal{F}^{(2)}$. Then the transience of $\mathbf{M}^{(1)}$ in the sense specified in Theorem 2.1(2) implies the transience of $\mathbf{M}^{(2)}$ in the same sense.*

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