Laplace's method for the laws of heat processes on loop spaces *

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- In this talk we will discuss Laplace's method for a class of heat processes on loop spaces. We will obtain the first term of the asymptotics under assumptions that the function under consideration attains its minimum at a unique point and that the Hessian at the point is non-degenerate. This kind of process was first introduced by P. Malliavin in 1990 for the loop group case and then gradually generalized by various authors. Our tool is the rough path theory of T. Lyons. This technique was pioneered by S. Aida for finite dimensional processes in his unpublished paper.

- We consider the following model. Let $\mathcal{L}_0^r$ be the r-dimensional pinned continuous loop space with the usual sup-norm. The pinned Wiener measure is supported on $\mathcal{L}_0^r$ and it satisfies the "exactness" condition, which implies the existence of Brownian rough paths. Let $P_0(\mathcal{L}_0^r)$ be the continuous path space over $\mathcal{L}_0^r$, which starts at 0. There exists $\mathcal{L}_0^r$-valued Brownian motion $(w_t)_{0\leq t\leq 1}$ associated with the pinned Wiener measure. It induces a probability measure on $P_0(\mathcal{L}_0^r)$. Let $\sigma_{ij}, \beta_i, b_i \in C_b^\infty(\mathbf{R}^d)$ $(1 \leq i \leq d, 1 \leq j \leq r)$. For each fixed "space parameter" $\tau \in S^1 \simeq [0,1]$, $(w_t(\tau))_{0\leq t\leq 1}$ is an r-dimensional (scaled) Brownian motion. We consider the following (finite dimensional) SDEs $(\varepsilon \in (0,1])$.

$$d_t X_t^\varepsilon(\tau) = \sigma(X_t^\varepsilon(\tau)) \circ \varepsilon d_t w_t(\tau) + b(X_t^\varepsilon(\tau))\varepsilon^2 dt + \beta(X_t^\varepsilon(\tau))dt \quad \text{with } X_0^\varepsilon(\tau) = 0.$$

  It is known that X^ε is a $P_0(\mathcal{L}_0^d)$-valued random variable. In this talk we discuus the Laplace's method for the law of X^ε.

- It is more natural to regard the above SDE as a Banach-valued differential equation in the rough path sense. Note that the rough path theory works on any Banach space, while the usual SDE theory does not work very well in a general Banach space. A large deviation principle for the law of X^ε (in fact, A large deviation principle for scaled Brownian rough paths) is also known. (For the ingredients of this paragraph, see [2].)

- Let $\mathcal{H}^r \subset P_0(\mathcal{L}_0^r)$ be the Cameron-Martin space for $P_0(\mathcal{L}_0^r)$. For $\gamma \in \mathcal{H}^r$, let $\phi = \Psi(\gamma) \in \mathcal{H}^d$ be the solution of the following ODE:

$$d_t\phi_t(\tau) = \sigma(\phi_t(\tau)) \cdot d_t\gamma_t(\tau) + \beta(\phi_t(\tau))dt \quad \text{with } \phi_0(\tau) = 0.$$

- The good rate function I for the large deviation principle for the laws of $\{X^\varepsilon\}_{\varepsilon>0}$ is given by

$$I(\phi) = \begin{cases} \frac{1}{2}\inf\{\|\gamma\|_{\mathcal{H}^r}^2 \mid \phi = \Psi(\gamma)\}, & \text{if } \phi = \Psi(\gamma) \text{ for some } \gamma \in \mathcal{H}^r, \\ \infty, & \text{otherwise.} \end{cases}$$

  It is well-known that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{E}\big[\exp(-F(X^\varepsilon)/\varepsilon^2)\big] = -\inf\{F(\phi) + I(\phi) | \phi \in P_0(\mathcal{L}_0^d)\}$$

  holds for a bounded and contiuous function $F : P_0(\mathcal{L}_0^r) \to \mathbf{R}$.

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- The aim of this talk is to investigate the exact asymptotics (not logarithmic asymptotics) of the integral on the left hand side of the above equation.

- **Assumption(A.0)** The function $F : P_0(\mathcal{L}_0^r) \to \mathbf{R}$ is a continuous and bounded.

  Assumption(A.1) The function $F \circ \Psi + \|\cdot\|_{\mathcal{H}^r}^2/2$ defined on $\mathcal{H}^r$ attains its minimum 0 at a unique point γ in $\mathcal{H}^r$. For this γ, we write $\phi = \Psi(\gamma)$.

  At the point γ of minimum we consider the Hessian $A := \nabla^2(F \circ \Psi)(\gamma)$. Note that we regard A as a bounded self-adjoint operator on $\mathcal{H}^r$ (or bounded symmetric bilinear form on $\mathcal{H}^r$). In fact, A will turn out to be a Hilbert-Schmidt operator.

  Assumption(A.2) The bounded continuous function F on $P_0(\mathcal{L}_0^r)$ is three times Fréchet differentiable on a neighborhood of γ and $\nabla^i F$ $(i = 1, 2, 3)$ is bounded on the neighborhood. The operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}^r}$ in the sense that $\inf\{(Ah, h)_{\mathcal{H}^r} | \|h\|_{\mathcal{H}^r} = 1\} > -1$.

- The following is the main result. It is possible to write the explicit value of α_0 in the following theorem, though we do not give it here.

  Theorem: *Let X^ε be as above and assume* **(A.0)**, **(A.1)**, *and* **(A.2)**. *Then, there is a constant α_0 such that*

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\big[\exp(-F(X^\varepsilon)/\varepsilon^2)\big] = \alpha_0.$$

- This technique is due to Aida. The keys are the rough path versions of (i) the stochastic Taylor expansion (cf. Azencott 1982), (ii) the Cameron-Martin theorem, and (iii) the Fernique theorem. With (i)–(iii), we can compute in the same way as in the case of (scaled) Gaussian measures on a Banach space.

References

[1] Inahama, Y.; Laplace's method for the law of heat process on loop spaces. Submitted. 44 pages.

[2] Inahama, Y.; Kawabi, H.; Large deviations for heat kernel measures on loop spaces via rough paths. Submitted. 22 pages.