Semiclassical problem of Schrödinger operators with variable coefficients on Wiener spaces

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1 Introduction

Let (B, H, μ) be an abstract Wiener space and set $\mu_\lambda(\cdot) = \mu(\sqrt{\lambda}\cdot)$. Let L_λ be the generator of the Dirichlet form on $L^2(B, d\mu_\lambda)$:

$$\mathcal{E}_\lambda(f, f) = \int_B |Df(w)|_H^2 d\mu_\lambda(w). \tag{1.1}$$

Let V be a real-valued measurable function on B. Let $\mathcal{E}_{\lambda,V}$ be a form which is defined by

$$\mathcal{E}_{\lambda,V}(f, f) = \mathcal{E}_\lambda(f, f) + \lambda^2 \int_B V(w)f(w)^2 d\mu_\lambda(w). \tag{1.2}$$

The Schrödinger operator $-L_{\lambda,V} = -L_\lambda + \lambda^2 V$ is the generator of the closure of the above form. In [1], we studied the asymptotic behavior of the lowest eigenvalue $E_0(\lambda)$ of $-L_{\lambda,V}$ as $\lambda \to \infty$. In this talk, we consider a closable form with variable coefficients on $L^2(B, \mu_\lambda)$:

$$\mathcal{E}_{\lambda,A}(f, f) = \int_B |A(w)Df(w)|_H^2 d\mu_\lambda(w) \tag{1.3}$$

and study the asymptotic behavior of the lowest eigenvalue of the Schrödinger operator $-L_{\lambda,A,V}$ which corresponds to the form:

$$\mathcal{E}_{\lambda,A,V}(f, f) = \mathcal{E}_{\lambda,A}(f, f) + \lambda^2 \int_B V(w)f(w)^2 d\mu_\lambda(w). \tag{1.4}$$

Here $A(w)$ is a bounded linear operator on H. In particular, we assume that $A(w) = I_H + T(w)$, where $T(w)$ is a trace class operator on H. This kind of Schrödinger operators naturally appear in the study of Schrödinger operators and Hodge-Kodaira type operators acting on differential forms on pinned path groups and submanifolds in Wiener spaces. In these examples, T and V are not continuous with respect to the topology of the Wiener space and not Fréchet differentiable. One way to avoid such a difficulty is to use rough path analysis. That study is still in progress and we mainly talk the case where T and V are Fréchet differentiable.

The key point to study the asymptotics of $E_0(\lambda)$ in [1] is the following Gross's Gaussian logarithmic Sobolev inequality: For $f \in \mathrm{D}(\mathcal{E}_\lambda)$,

$$\mathrm{Ent}_{\mu_\lambda}(f^2) := \int_B f(w)^2 \log\left(f(w)^2/\|f\|_{L^2(B,\mu_\lambda)}^2\right) d\mu_\lambda(w) \quad \leq \quad \frac{2}{\lambda}\mathcal{E}_\lambda(f, f). \tag{1.5}$$

If $A(w)$ is uniformly elliptic, that is, $\kappa = \inf_{w,h,\|h\|_H=1} \|A(w)h\|_H > 0$, then (1.5) implies a logarithmic Sobolev inequality:

$$\mathrm{Ent}_{\mu_\lambda}(f^2) \quad \le \quad \frac{2}{\lambda\kappa^2}\mathcal{E}_{\lambda,A}(f,f) \quad f \in \mathrm{D}(\mathcal{E}_\lambda). \tag{1.6}$$

However this inequality is not useful for our study in general and we use different inequality. To this end, we assume the following strong assumption on T:

(A1) There exists a trace class operator T_0 such that for any w and $h \in H$, $\|(I_H + T(w))h\|_H \ge \|(I_H + T_0)h\|_H > 0$.

Under (A1) and some integrability assumptions, we prove the following estimate:

$$\begin{aligned} E_0(\lambda) \quad &\ge \quad -\frac{\lambda}{2}\log\left[\int_B \exp\Big\{-2\lambda V_T(w) + T_1(w)\Big\} d\mu_\lambda(w)\right] \\ &\qquad +\frac{\lambda}{2}\log\left(\det(I_H + T_0)\right), \end{aligned} \tag{1.7}$$

where

$$V_T(w) \quad = \quad V(w) + \frac{1}{4}\|T(w)w\|_H^2 + \frac{1}{2}(T(w)w, w)_H \tag{1.8}$$

$$\begin{aligned} T_1(w) \quad &= \quad \mathrm{tr}\Big[T(w)^*T(w) + T(w) + T(w)^* \\ &\qquad + D.(T(w)^*T(w))w + (D.T)(w))w + (D.T^*)(w)w\Big]. \end{aligned} \tag{1.9}$$

$(T(w)w, w), \|T(w)w\|_H^2, T_1(w)$ are not well-defined just on the assumptions that T is a trace class. So we put much stronger assumptions on T which we explain in the talk. Note that (1.7) is equivalent to a logarithmic Sobolev inequality "with a potential function" on B which is different from (1.6). The estimate (1.7) is used to obtain the lower bound estimate of the asymptotic behavior of $\lim_{\lambda\to\infty} \frac{E_0(\lambda)}{\lambda}$.

For the study of Schrödinger operators on path spaces over Riemannian manifolds, we need to consider a bad coefficient $A(w)$ which are neither bounded nor continuous in w. If time permits, we make some remarks on such problems also.

References

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