Schrödinger operators on the Wiener space*

Ichiro SHIGEKAWA† (Kyoto University)

We consider a Schrödinger operator $A = L - V$ on an abstract Wiener space (B, H, μ). Here L is the Ornstein-Uhlenbeck operator and V is a scalar potential. Our goal is to determine the domain of $L - V$. To be precise, we will show that $\mathrm{Dom}(A) = \mathrm{Dom}(L) \cap \mathrm{Dom}(V)$ under a suitable condition. We also show the spectral of gap of this operator.

Essential self-adjointness

We define $\mathcal{F}C_0^\infty$ to be the set of all functions $f\colon B \to \mathbb{R}$ such that there exist $n \in \mathbb{N}$, $F \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_1, \ldots, \varphi_n \in B^*$ with

$$f(x) = F(\langle x, \varphi_1\rangle, \ldots, \langle x, \varphi_n\rangle). \tag{1}$$

A Schrödinger operator $L - V$ is defined on $L^2(\mu)$. Essentially self-adjointness of $L - V$ is a fundamental problem and we have the following theorem:

Theorem 1. For a Schödinger operator $L - V$ on an abstract wiener space (B, H, μ), we suppose that V_+, $e^{V_-} \in L^{2+}$ where $V_+ := \max\{V, 0\}$, $V_- := \max\{-V, 0\}$. Then $L - V$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$.

To show this theorem, we need the followig result for the logarithmic Sobolev inequalities. In general setting, the (defective) logarithmic Sobolev inequality for a Dirichlet form $\mathcal{E}$ is written as

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \alpha\mathcal{E}(f, f) + \beta\|f\|_2^2. \tag{2}$$

Here (B, μ) is a general measure space. We also denote the associated generator by L (not specify to the Ornstein-Uhlenbeck operator). For the Dirichlet form $\mathcal{E}$, we assume the local property and the existence of square field opetator, i.e.,

$$\mathcal{E}(f, g) = \int_B \Gamma(f, g)\, d\mu \tag{3}$$

and Γ has the derivation property. In the case of an abstract Wiener space, $\Gamma(f, f) = |\nabla f|^2$, ∇ being a gradient operator. We have the following theorem.

Theorem 2. Asume that the logarithmic Sobolev (2) holds. Then, for any $\varepsilon > 0$, there exist positive constants K_1, K_2 such that

$$\int_B f^2 \log_+^2 f\, d\mu \le \alpha^2(1 + \varepsilon)\|Lf\|_2^2 + K_1 + K_2\|f\|_2^6. \tag{4}$$

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†E-mail: `ichiro@math.kyoto-u.ac.jp` URL: `http://www.math.kyoto-u.ac.jp/~ichiro`

The domain of a Schrödinger operator

We consider an issue of the domain of a Schrödinger operator of the form $A = L - V + W$. Here we decompose the potential as follows:

(A.1) $V \geq 1$ and $V \in L^{2+}$.

(A.2) W is non-positive and there exists a constant $0 < \alpha < 1$ such that $e^W \in L^{2/\alpha}$.

Tough there are many ways to give sufficient conditions, we restrict ourselves to typical ones. One of them is

$$e^{W+|b|^2} \in L^{2/\alpha}. \tag{5}$$

The other is that there exists a constant $C > 0$ such that

$$|b|^2 \leq \alpha V + C. \tag{6}$$

Then we have the following

Theorem 3. We assume the same assumptions as before. Then we have that $\mathrm{Dom}(A) = \mathrm{Dom}(L) \cap \mathrm{Dom}(V)$. Moreover, for sufficiently large λ, there exist positive constants K_1, K_2 such that

$$K_1 \|(A-\lambda)f\|_2 \leq \|Lf\|_2 + \|Vf\|_2 \leq K_2 \|(A-\lambda)f\|_2. \tag{7}$$

Spectral gap of a Schrödinger oerator

We denote the set of spectrum of $A = L - V + W$ by $\sigma(A)$ and set $l = \sup \sigma(A)$. Then, using Theorem 3, we have

Theorem 4. Assume that V, W sasisfy the conditions (A.1), (A.2). We also assume that either (5) or (6) is fulfilled. Then the spectrum of $A = L - V + W$ is discrete on $(l-1, l]$, i.e., it consists of point spectrums of finite multiplicity.

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