STOCHASTIC NONLINEAR BEAM EQUATION

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ABSTRACT

Suppose H is a real separable Hilbert space and let A and B be self-adjoint operators in H. Suppose that $B > 0$ and that $A \geq \mu I$ for some $\mu > 0$. We assume that $D(A) \subset D(B)$ and that $B \in \mathcal{L}(D(A), H)$ with $D(A)$ endowed with the norm $\|x\|_{D(A)} := \sqrt{\langle Ax, Ax\rangle}$. Suppose G is another real separable Hilbert and that $W(t)$, $t \geq 0$ is an G-cylindrical Wiener process on some probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$. Suppose that $m : [0, \infty) \to \mathbb{R}$ is a nonnegative C^1-class function. Finally, $f : D(A) \times H \to H$ is locally Lipschitz (i.e. Lipschitz on balls) and satisfies the following one sided linear growth condition (for some $L > 0$)

$$\langle y, f(x, y)\rangle \geq -L(1 + |x|^2_{D(A)} + |y|^2), \quad (x, y) \in D(A) \times H. \tag{1.1}$$

We also consider a locally Lipschitz and of linear growth map $\sigma : D(A) \times H \to \mathcal{L}_2(G, H)$, where $\mathcal{L}_2(G, H)$ is the space of all Hilbert-Schmidt operators from G to H. Then, given $(x_0, x_1) \in D(A) \times H$ we consider the following second order stochastic differential equation

$$\begin{aligned} x_{tt} + A^2x + f(x, x_t) &= m(|B^{1/2}x|^2)Bx = \sigma(x, x_t)dW, \\ x(0) &= x_0, \;\; x_t(0) = x_1. \end{aligned} \tag{1.2}$$

Problem (1.2) can be rewritten as a stochastic evolution equation (of first order) in the Hilbert space $\mathcal{H} = D(A) \times H$ (endowed with a natural scalar product)

$$du = (\mathcal{A}u + F(u))\, dt + \Sigma(u)\, dW(t), \;\; u(0) = u_0, \tag{1.3}$$

where $u_0 = (x_0, x_1)$ and with, informally $u = (x, x_t)$. Here, as in the deterministic theory of second order equations, the operator $\mathcal{A}$ in $\mathcal{H}$ is defined by $\mathcal{U}(x, y) = (y, -A^2x)$, $F : \mathcal{H} \to \mathcal{H}$ is defined by $F(x, y) = \big(0, -m(|B^{1/2}x|^2)Bx - f(x, y)\big)$ and $\Sigma : \mathcal{H} \to \mathcal{L}_2(G, \mathcal{H})$ is defined by $\Sigma(x, y)(g) = (0, \sigma(x, y)g)$. With $D(\mathcal{A}) = D(A^2) \times D(A)$ the operators $\mathcal{U}$ and $-\mathcal{U}$ are m-dissipative. By a solution $x(t)$ of the problem (1.2) we mean the first component of the solution $u(t)$ to the problem (1.3). Our main results are summarised in the following

Theorem 1.1. *Set $M(s) = \int_0^s m(r)\, dr$, $s \geq 0$. If the initial data u_0 is such that*

$$\mathbb{E}\left(|u_0|^2 + M(|B^{1/2}u_0|^2)\right) < \infty, \tag{1.4}$$

then there exists a unique global mild solution to the problem (1.3). The paths of this solutions are continuous ($\mathcal{H}$-valued) a.s.
Moreover, with some $C > 0$,

$$\mathbb{E}\left(|u(t)|^2 + M(|B^{1/2}u(t)|^2)\right) \leq e^{Ct}\big(2 + \mathbb{E}\left(|u_0|^2 + M(|B^{1/2}u_0|^2)\right)\big), \quad t \geq 0.$$

Theorem 1.2. *Under some additional conditions, in particular that the damping term g is of the form $g(x, y) = \beta y$, for all $(x, y) \in \mathcal{H}$ and some $\beta > 0$, and that $ym(y) \geq \alpha M(y)$, for all $y \geq 0$ and some $\alpha > 0$, the zero solution to problem (1.3) is exponentially mean-square stable and exponentially stable with probability one. To be precise, there exist constants $C < \infty$, $\lambda > 0$ such*

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that for any solution u *is a to problem (1.3) with the initial data* u_0 *satisfying condition (1.4), we have:*

$$\mathbb{E}\|u(t)\|^2_{\mathcal{H}} \leq Ce^{-\lambda t}\mathbb{E}\left(|u_0|^2 + M(|B^{1/2}u_0|^2)\right) \quad \textit{for all } t \geq 0,$$

and for every $\lambda^* \in]0,\lambda[$ *we can find a function* $t_0 : \Omega \longrightarrow [0,\infty)$ *such that*

$$\|u(t)\|^2_{\mathcal{H}} \leq Ce^{-\lambda^* t}\mathcal{E}(u_0) \quad \textit{for all } t \geq t_0, \mathbb{P}\textit{-almost surely}.$$

The proof of Theorem 1.1 is in some way standard. We first prove existence of a maximal local mild solution $u(t)$, $t \in [0,\tau)$, , see also [2]. Next we prove that this solution is global (i.e. $\tau = \infty$) by using the Khasminski test of non-explosion with the Lyapunov function $V : \mathcal{H} \to \mathbb{R}$ defined by $V(u) = \frac{1}{2}\left(|u|^2 + M(|B^{1/2}x|^2\right)$, where $u = (x,y)$. On an informal level the last part is trivial. However, since firstly $u(t)$ is not a semimartingale, and secondly, because of some peculiarity of the nonlinear term, the full proof requires some delicate approach. We also study Feller property of the process $u(t)$ and we plan to investigate existence of an invariant measure. In order to prove Theorem 1.2 we use another Lyapunov function:

$$\Phi(u) = \|Ax\|^2 + \|y\|^2 + \|\beta x + y\|^2 + \frac{1}{2}M(\|B^{1/2}x\|^2), \quad u = (x,y) \in \mathcal{H}.$$

Example 1.3. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ with (sufficiently) smooth boundary. Let $H = L^2(\mathcal{O})$, let B be the $-$Laplacian with Dirichlet boundary conditions: $D(B) = H^{2,2}(\mathcal{O}) \cap H^{1,2}_0(\mathcal{O})$. Define a self-adjoint operator C by $D(C) = \left\{\psi \in H^{4,2}(\mathcal{O}) : \psi = \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\mathcal{O}\right\}$, $Cu := \Delta^2 u$. Define finally $A = C^{1/2}$. Then our problem (1.2) becomes the following stochastic beam equation

$$u_{tt} - m\left(\int_{\mathcal{O}} |\nabla u|^2\,dx\right) + \gamma\Delta^2 u + f(x,u,\nabla u,u_t) = \pi(x,u,\nabla u,u_t)\dot{W} \tag{1.5}$$

with the clamped boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\mathcal{O} \tag{1.6}$$

Example 1.4. Equation (1.5) can also be studied with so called hinged boundary conditions

$$u = \Delta u = 0 \text{ on } \partial\mathcal{O} \tag{1.7}$$

The problem (1.5), (1.7) is also of the form (1.2) with A defined to be equal to B. The coefficient $f : D := \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz in the last three variables, $f(\cdot,0,0,0) \in L^2(\mathcal{O})$ and

$$f(x,r,s,z)z \geq -L(1+|z|^2),$$

for some $L \geq 0$ and all $(x,r,s,z) \in D$. There are also some natural assumptions on the diffusion coefficient π and on the Hilbert space G.

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