Hypoellipticity in infinite dimensions

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Let (B, H, μ) be an abstract Wiener space. We consider the following type (infinite dimensional) SDE's on (B, H, μ):

$$dX_t = dW_t + A(X_t)dW_t + b(X_t)dt \tag{1}$$

with $X_0 = 0$, where W_t is a B-*valued Wiener process* and $A : B \to H \otimes H$, $b : B \to H$ are measurable maps. In [1], the author showed the absolute continuity of the distribution of X_t in uniformly elliptic case (Theorem 1 and Theorem 2 stated below). In this talk, we discuss more general hypoelliptic case.

Let $\mathbf{W} = \{w \in C([0,T] \to B); w_0 = 0\}$, $\mathbf{H} = \{\mathbf{h} \in \mathbf{W}; \int_0^T |\dot{\mathbf{h}}(t)|_H^2 dt < \infty\}$ and P be a standard Wiener measure on $\mathbf{W}$. The triple $(\mathbf{W}, \mathbf{H}, P)$ is also an abstract Wiener space. The modified Malliavin covariance $\sigma(t) \in H \otimes H$ is defined by

$$\langle (tI_H + \sigma(t))h, g\rangle_H = \Big(D\langle X_t, h\rangle_H, D\langle X_t, g\rangle_H\Big)_{\mathbf{H}}, \quad h, g \in H,$$

where D is the $\mathbf{H}$-derivarive.

Let E and F be separable Hilbert spaces. $L(E; F)$ denotes the Banach space consisting of bounded linear operators from E to F. $L^k_{(2)}(E; F)$ denotes the Hilbert space consisting of Hilbert-Schmidt multi-linear operators from $\underbrace{E \times \cdots \times E}_{k}$ to F. We often identify $L_{(2)}(E; F) = L^1_{(2)}(E; F)$ with $E \otimes F$.

H-Fréchet differentiability

Let E be a separable Hilbert space. We say that a map $f : B \to E$ is *continuously H-Fréchet differentiable* if there exists a continuous map $f^{(1)} : B \to L_{(2)}(H; E)$ such that

$$\lim_{|h|_H \to 0} \frac{|f(x+h) - f(x) - f^{(1)}(x)h|_E}{|h|_H} = 0$$

for each $x \in B$. We can define inductively n-times continuously H-Fréchet differentiability and n-times continuously H-Fréchet derivative $f^{(n)}$ by $f^{(n)} = (f^{(1)})^{(n-1)}$, $n = 2, 3, \ldots$. We denote by $\mathcal{CH}_b^\infty(E)$ the collection of infinitely many times continuously H-Fréchet differentiable functions $f : B \to E$ such that $\sup_{x \in B} |f^{(n)}(x)|_{L^n_{(2)}(H;E)} < \infty$ for all $n \in \mathbf{Z}_+$.

Theorem 1. *Assume that $A \in \mathcal{CH}_b^\infty(H \otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$ and*

$$E\Big[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}\Big] < \infty, \quad \text{for all } p \in (1, \infty). \tag{2}$$

Then the distribution of X_t is absolutely continuous with respect to μ_t(the distribution of $x \in B \mapsto \sqrt{t}x \in B$ under μ).

Theorem 2. *Let* $A \in \mathcal{CH}_b^\infty(H \otimes H)$ *and* $b \in \mathcal{CH}_b^\infty(H)$. *If* $\inf_{|h|_H=1} |(I_H + A(0))h|_H > 0$, *then* (2) *holds.*

Trace class

Let E be a separable Hilbert space. We say that a bounded bilinear operator $T : H \times H \to E$ is in $\mathcal{T}(E)$ if

$$|T|_{\mathcal{T}(E)} = \sup \sum_{i=1}^{\infty} |T(e_i, f_i)|_E < \infty,$$

where the supremum is taken over all CONS's $\{e_i\}$ and $\{f_i\}$ in H. For $F \in L_{(2)}^n(H, E)$, $n \geq 2$, define $F_{[2]} : H \times H \to L_{(2)}^{n-2}(H; E)$ by $F_{[2]}(h, g) = F[h, g, \cdot \cdots, \cdot]$. We denote by $\mathcal{TH}_b^\infty(E)$ the collection of $f \in \mathcal{CH}_b^\infty(E)$ such that, for every $x \in B$ and $n = 2, 3, \ldots$, the map $f_{[2]}^{(n)}(x) : H \times H \to L_{(2)}^{n-2}(H; E)$ is in $\mathcal{T}(L_{(2)}^{n-2}(H; E))$ and

$$\sup_{x \in B} |f_{[2]}^{(n)}(x)|_{\mathcal{T}(L_{(2)}^{n-2}(H;E))} < \infty.$$

Hypoellipticity

For $U, V \in \mathcal{CH}_b^\infty(H)$, the *Lie bracket* $[U, V] \in \mathcal{CH}_b^\infty(H)$ is defined by

$$[U, V](x) = V^{(1)}(x)[U(x)] - U^{(1)}(x)[V(x)], \quad x \in B.$$

Fix a CONS $\{e_i\}$ in H. Let $V_i(x) = e_i + A(x)e_i$. For $j \in \mathbf{N}$, define

$$\Sigma_j = \{[V_{i_1}, [V_{i_2}, \cdots, [V_{i_{j-1}}, V_{i_j}] \cdots]]; i_1, i_2, \ldots, i_j = 1, 2, \cdots\}$$

and $\tilde{\Sigma}_j = \bigcup_{i=1}^{j} \Sigma_i$.

Theorem 3. *Assume that* $A \in \mathcal{TH}_b^\infty(H \otimes H)$ *and* $b \in \mathcal{CH}_b^\infty(H)$. *Assume also that there exists* $N \in \mathbf{N}$ *such that the closure of the linear subspace of* H *spanned by* $\{V(0); V \in \tilde{\Sigma}_N\}$ *coincides to* H. *Then* (2) *holds.*

References

[1] Heya, N., The absolute continuity of a measure induced by infinite dimensional stochastic differential equations, to appear in J. Fac. Sci. Univ. Tokyo.