KATO CLASS MEASURES UNDER HEAT KERNEL ESTIMATE

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1. Kato class

Let (X, d) be a proper locally compact separable metric space and m a positive Radon measure with full support, where proper means that every closed ball is compact. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(X; m)$ and $\mathbf{M}= (\Omega, X_t, P_x)$ the m-symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$. We assume the existence of the heat kernel $p_t(x, y)$ of $\mathbf{M}$ and $p_t(x, y)$ is defined for all $(t, x, y) \in]0, \infty[\times X \times X$. Fix $\nu > 0$. We assume $(E\Phi_\beta)$ for $\beta > 0$: there exist positive constants $C_1 < C_2$ and $t_0 > 0$ such that for all $(x, y) \in X \times X$ and $t \in]0, t_0[$

$$\frac{C_1}{t^{\nu/\beta}}\Phi_1\Big(\frac{d(x,y)}{t^{1/\beta}}\Big) \leq p_t(x,y) \leq \frac{C_2}{t^{\nu/\beta}}\Phi_2\Big(\frac{d(x,y)}{t^{1/\beta}}\Big), \tag{1}$$

where Φ_i $(i = 1, 2)$ are positive decreasing functions on $[0, \infty[$ such that $\displaystyle\int_1^\infty t^{\nu-1}\Phi_2(t)dt < \infty$.

A positive Borel measure μ on X is said to be in the *Kato class* K_ν if

$$\lim_{r\to 0}\sup_{x\in X}\int_{B_r(x)}\frac{\mu(dy)}{d(x,y)^{\nu-\beta}} = 0 \text{ if } \nu > \beta \tag{2}$$

$$\lim_{r\to 0}\sup_{x\in X}\int_{B_r(x)}(\log d(x,y)^{-1})\mu(dy) = 0 \text{ if } \nu = \beta \tag{3}$$

$$\sup_{x\in X}\int_{B_1(x)}\mu(dy) < \infty \text{ if } \nu < \beta. \tag{4}$$

By definition, we see that every $\mu \in K_\nu$ is a Radon measure. On the other hand, we have another definition of Kato class measure in terms of $p_t(x, y)$: A positive Borel measure μ on X is said to be in S_K^0 (resp. S_D^0) if

$$\lim_{t\to 0}\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy) = 0. \tag{5}$$

$$(\text{resp. } \sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy) < \infty \text{ for some/all } t > 0). \tag{6}$$

Let S_1 be the family of smooth measures in the strict sense (see [3] for the definition of S_1). We set $S_K := S_K^0 \cap S_1$ (resp. $S_D := S_D^0 \cap S_1$). We say that $\mu \in S_K^{loc}$ (resp. S_D^{loc}) if and only if $I_A\mu \in S_K^0$ (resp. $I_A\mu \in S_D^0$) for any Borel set A with $\mu(A) < \infty$. We see that every finite measure $\mu \in S_D^0$ is in S_{00}, that is, such μ is a measure of finite energy with bounded potential, hence if $\mu \in S_D^{loc}$ is a Radon measure, then $\mu \in S_1$, consequently, there exists a positive continuous additive functional A_t (PCAF in short) of $\mathbf{M}$ admitting no exceptional set associated with any Radon measure $\mu \in S_D^{loc}$ (cf. [3]). Hence (6) is equivalent to $\sup_{x \in X} E_x[A_t] < \infty$ for some $t > 0$ in that case.

Our result is the following:

Theorem 1.1. *$S_K^0 \subset K_\nu \subset S_D^{loc}$. In particular, $K_\nu \subset S_1$.*

Theorem 1.2. *Suppose that $\mathbf{M}$ is conservative and m satisfies that there exists $V > 0$ such that $\sup_{x \in X} m(B_r(x)) \le V r^\nu$, $\forall r > 0$. Then $S_K = S_K^0 = K_\nu$.*

In Aizenman-Simon [1] or Chung-Zhao [2], $K_d = S_K$ is showed in the case $\mu(dx) = |f(x)|dx$, $f \in K_d$ for Brownian motion on $\mathbb{R}^d$. Zhao [6] extends this in more general setting including a subclass of Lévy processes, but his result assures the case $d \ge 2$ for Brownian motion and the case $d > \alpha$ for symmetric α-stable process on $\mathbb{R}^d$. Our result extends [1] and can be applicable to various settings including the case of diffusion processes on fractals.

2. Example

Example 2.1 (Symmetric α-stable process)**.** Take $\alpha \in]0, 2[$. Let $\mathbf{M}^\alpha = (\Omega, X_t, P_x)_{x \in \mathbb{R}^d}$ be the symmetric α-stable process on $\mathbb{R}^d$, that is, Lévy process satisfying $E_0[e^{\langle \xi, X_t \rangle}] = e^{-t|\xi|^\alpha}$. $\mathbf{M}^\alpha$ admits a semigroup kernel $p_t(x, y)$ satisfying the following estimate (see [4]):

$$\frac{C_1}{t^{d/\alpha}} \frac{1}{\left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}} \le p_t(x, y) \le \frac{C_2}{t^{d/\alpha}} \frac{1}{\left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}}. \tag{7}$$

Then $K_d = S_K$, in particular, for surface measure σ on S^{d-1}, we have $\sigma \in S_K$ if and only if $\alpha > 1$.

References

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