Local B-model and Mixed Hodge Structure

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Remote B-model and MHS

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Motivation

Jacobian ring description of $H^2(\mathbb{T}^2, C_{a}^o)$

Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_{a}^o)$

Yukawa coupling

Holomorphic anomaly equation

Witten’s Geometric Quantization Approach

Appendix (Examples etc.)

Remark

In this talk:

- All manifolds (varieties) are complex.
- All variables and parameters are complex.
- $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$ $n$-dimensional complex torus, not $(S^1)^n.$
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2 Jacobian ring description of $H^2(\mathbb{T}^2, C_a^\circ)$

3 Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_a^\circ)$

4 Yukawa coupling

5 Holomorphic anomaly equation

6 Witten’s Geometric Quantization Approach

7 Appendix (Examples etc.)
Local Mirror Symmetry

- Local mirror symmetry is a variant of (ordinary) mirror symmetry.
- It is derived from mirror symmetry of toric Calabi–Yau hypersurfaces by considering a certain limits in moduli spaces. (e.g. CY hypersurface $\subset \hat{\mathbb{P}}(1, 1, 1, 6, 9) \sim \mathbb{P}^2$) [Katz–Klemm–Zaslow (1997), Chiang–Klemm–Yau–Zaslow (1999)]
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Local Mirror Symmetry

A-hypergeometric system with $\beta = \vec{0}$

Solutions give:
- Mirror map
- A derivative of prepotential

$\Delta$

2 dim reflexive polyhedron

PF equation for period integrals of "top element"

$\left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$

Compact toric surface

$\mathbb{P}$ s.t. $-K_{\mathbb{P}}$ nef

$g = 0$ local GW inv.

Local A-model

A family of affine curves

$C_a \subset \mathbb{T}^2$

VMHS on $H^2(\mathbb{T}^2, C_a^0)$

Local B-model

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Local B-model and MHS
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A

MS

GW inv of $X$

B

VHS on $H^3(X^\vee)$

• holo. 3-form $\Omega$

• Yukawa coupling

LMS

local GW inv of $\mathbb{P}$

VMHS on $H^2(\mathbb{T}^2, C_a^0)$

• $\omega := (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$

• ?? $\leftrightarrow$ This talk

Important because it is:

• a third derivative of prepotential;

• necessary for BCOV’s holomorphic anomaly eq.
Comparison with Mirror Symmetry

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\[ \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega \]

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- GW inv of $\mathcal{X}$
- $\text{VHS on } H^3(\mathcal{X}^\vee)$
  - holomorphic 3-form $\Omega$
  - Yukawa coupling

- $\int_{\mathcal{X}^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega$

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- local GW inv of $\mathbb{P}$
- $\text{VMHS on } H^2(\mathbb{T}^2, C_{\hat{a}})$
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Our aim

- In several examples of local B-model, the Yukawa couplings have been computed [Klemm–Zaslow, Jinzenji–Forbes, Aganagic–Bouchard–Klemm, Haghihat–Klemm–Rauch, Alim–Länge-Mayr, Brini–Tanzini]. However, there has been no direct definition.

- We gave a definition of local B-model Yukawa coupling using the results of Batyrev, Stienstra on the VMHS on $H^2(\mathbb{T}^2, C_a^o)$.

- We also proposed how to modify Bershadsky–Cecotti–Ooguri–Vafa’s holomorphic anomaly equation to the setting of local B-model.
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5. Holomorphic anomaly equation

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Description of $H^2(\mathbb{T}^2, C_a^\circ)$

- $H^2(\mathbb{T}^2, C_a^\circ)$ was studied by Batyrev (’93) and Stienstra (’97).
- $H^2(\mathbb{T}^2, C_a^\circ)$ has a Jacobian-ring like description. It is isomorphic to a (quotient) vector space $\mathcal{R}_{F_a}$, which is determined by the data of $\Delta$ and $F_a(t)$.
- The mixed Hodge structure on $H^2(\mathbb{T}^2, C_a^\circ)$ is given in terms of filtrations of $\mathcal{R}_{F_a}$.
- The variation of mixed Hodge structures on $H^2(\mathbb{T}^2, C_a^\circ)$ is also described in terms of $\mathcal{R}_{F_a}$ ($\nabla a_m \iff$ Derivation by $a_m$ on $\mathcal{R}_{F_a}$).
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"Jacobian Ring" $\mathcal{R}_{F_a}$

- $\Delta(k)$: the polyhedron obtained by enlarging $\Delta$ by $k$-times.  
- Consider the (infinite dim) vector space spanned by monomials $t_0^k t_m$ "lying on $\Delta(k)$":

$$S_{\Delta}^k := \bigoplus_{m \in \Delta(k)} \mathbb{C} t_0^k t_m \quad (t_m := t_1^m t_2^m)$$

$$S_{\Delta} := \bigoplus_{k \geq 0} S_{\Delta}^k, \quad \deg t_0^k t_m := k \quad \text{(a graded ring)}$$

- Define the differential operators on $S_{\Delta}$: $(\theta_x := x \partial_x)$

$$\mathcal{D}_0(t_0^k t_m) = (k + t_0 F_a(t)) t_0^k t_m$$

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- The "Jacobian ring" $\mathcal{R}_{F_a}$ is the quotient vector space:

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Examples of $\mathcal{R}_F$

• $\Delta = \triangle \Rightarrow F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}$

Relations $\mathcal{D}_i 1 = 0 \ (i = 0, 1, 2)$ imply:

$$t_0 t_1 = -\frac{a_0}{3a_1} t_0, \quad t_0 t_2 = -\frac{a_0}{3a_2} t_0, \quad \frac{t_0}{t_1 t_2} = -\frac{a_0}{3a_3} t_0.$$ 

By similar calculation, $t_0^k t_1^m \ (k \geq 2)$ is equal to

$$\text{const.} t_0^2 + \text{term of } t_0\text{-degree 1}.$$  

$\therefore \mathcal{R}_F \cong \mathbb{C}1 + \mathbb{C}t_0 + \mathbb{C}t_0^2$

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$$\therefore \mathcal{R}_{F_a} \cong \mathbb{C} 1 \oplus \mathbb{C} t_0 \oplus \mathbb{C} t_0^2$$

- $\Delta = \begin{array}{c} \triangle \end{array}$
\[ \mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C_a^\circ) \]

- For a (general) 2-dim reflexive polyhedron \( \Delta \),
  \[ \mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus R^1_{F_a} \oplus \mathbb{C}t_0^2 \quad (\text{dim} = \# \Delta \cap \mathbb{Z}^2 - 1) \]
  \[ R^1_{F_a} := S^1_\Delta / \mathbb{C}t_0 F_a \oplus \mathbb{C}t_1 \theta_t F_a \oplus \mathbb{C}t_2 F_a \quad \text{(degree 1-part)} \]

- Note that there is an exact sequence
  \[
  0 \longrightarrow PH^1(C_a^\circ) \longrightarrow H^2(\mathbb{T}^2, C_a^\circ) \longrightarrow H^2(\mathbb{T}^2) \longrightarrow 0 \\
  (PH^1(C_a^\circ) := H^1(C_a^\circ)/H^1(\mathbb{T}^2))
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  \[
  \mathbb{C}1 \leftrightarrow H^2(\mathbb{T}^2) : 1 \leftrightarrow \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \\
  R^1_{F_a} \oplus \mathbb{C}t_0^2 \leftrightarrow PH^1(C_a^\circ) : t_0^k t \leftrightarrow \left( 0, \text{Res}_{F_a=0} \frac{(k - 1)!t^m}{(-1)^{k-1} F_a^k} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \right)
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  \]
  \[ R^1_{F_a} := S_{\Delta}^1 \otimes \mathbb{C} t_0 F_a \oplus \mathbb{C} t_1 \theta_1 F_a \oplus \mathbb{C} \theta_2 F_a \quad \text{degree 1-part} \]

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  \]
Examples of $\mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C_a^\circ)$

- $C_a :=$ compactification of $C_a^\circ$, genus $C_a = 1$. 

- $\Delta = \triangle$ 

\[ \mathcal{R}_{F_a} \cong \mathbb{C} \bigoplus \mathbb{C} t_0 \bigoplus \mathbb{C} t_0^2 \]
\[ \begin{align*}
\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \\
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\end{align*} \]
\[ (1, 0)-\text{form on } C_a \]
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Examples of $\mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C_a^\circ)$

- $C_a :=$ compactification of $C_a^\circ$, genus $C_a = 1$.

- $\Delta$ = \includegraphics[width=.5\textwidth]{example_triangle}

$$\mathcal{R}_{F_a} \cong \begin{array}{c}
\mathbb{C}1 \\ \oplus \\
\mathbb{C}t_0 \\ \oplus \\
\mathbb{C}t_0^2 \\
\end{array}$$

\[
\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}
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Outline

1. Motivation
2. Jacobian ring description of $H^2(\mathbb{T}^2, C_a^\circ)$
3. Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_a^\circ)$
4. Yukawa coupling
5. Holomorphic anomaly equation
6. Witten's Geometric Quantization Approach
7. Appendix (Examples etc.)
What’s Mixed Hodge Structure?

- \( H^k(V) \) of a smooth projective variety \( V \) has the canonical Hodge structure of weight \( k \):

\[
H^k(V) = \bigoplus_{p+q=k} H^{p,q}(V) \quad \text{(Hodge decomposition)}
\]

- To generalize this to \( H^k(U) \) of an open variety \( U \), it is necessary to consider the mixed Hodge structure.

- Roughly speaking, the mixed Hodge structure is the direct sum of Hodge structures of different weights:

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\bigoplus_{l} \bigoplus_{p+q=l} H^{p,q}
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Definition of MHS

- Mixed Hodge structure of weight $k$ consists of:
  - free abelian group $H^*_\mathbb{Z}$,
  - the weight filtration $W_\bullet$ on $H^*_\mathbb{Z}$ (increasing filtration),
  - the Hodge filtration $F^\bullet$ on $H^*_\mathbb{C}$ (decreasing filtration),

such that the induced Hodge filtration on $W_I/W_{I-1}$ has a Hodge structure of weight $I + k$.

$$H^{p,k+I-p} := \frac{F^p W_I/W_{I-1}}{F^{p+1} W_I/W_{I-1}}$$

satisfy $H^{p,q} = H^{q,p}$. 
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\[
H^{p,k+l-p} := \frac{F^p W_i / W_{i-1}}{F^{p+1} W_i / W_{i-1}} \quad \text{satisfy} \quad H^{p,q} = \overline{H^{q,p}}.
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MHS for an open variety

- If $U = V - D$ where $V$ is a smooth projective variety and $D$ is a simple normal crossing divisor, then $H^k(U)$ has a canonical mixed Hodge structure.

- Hodge filtration $F^\bullet$ is induced from the filtration on $\Omega^\bullet_V(\log D)$

$$F^p \Omega^\bullet_V(\log D) = \Omega^{\geq p}_V(\log D)$$

- Weight filtration is induced from the filtration

$$W_l \Omega^\bullet_V(\log D) = \wedge^l \Omega^1_V(\log D) \wedge \Omega^{\leq -l}_V.$$

Roughly speaking, $W_{k+l} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

- For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

$$\ldots \rightarrow H^k(U_1) \rightarrow H^{k+1}(U_2, U_1) \rightarrow H^{k+1}(U_2) \rightarrow \ldots$$

is a long exact sequence of MHS’s.
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Filtrations on $H^2(\mathbb{T}^2, C_\alpha^\circ)$

- **Hodge filtration:**
  Let $\mathcal{E}^{-i}$ ($i = 0, 1, 2, \ldots$) be the subspace of $\mathcal{R}_F$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.

  \[
  0 \subset \mathcal{E}^0 = \mathbb{C}1 \subset \mathcal{E}^{-1} \subset \mathcal{E}^{-2} = \mathcal{R}_F
  \]

  \[
  0 \subset F^2 \subset F^1 \subset F^0 = H^2(\mathbb{T}^2, C_\alpha^\circ)
  \]

- **Weight filtration:**
  Let $\mathcal{I}_j$ ($1 \leq j \leq 3$) $\subset \mathcal{R}_F$ be spanned by the images of $t_0^k t^m$'s such that $k \geq 1$, $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $\mathcal{I}_4 := \mathcal{R}_F$.

  \[
  0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \subset \mathcal{I}_4 = \mathcal{R}_F
  \]

  \[
  0 \subset W_1 \subset W_2 = W_3 \subset W_4 = H^2(\mathbb{T}^2, C_\alpha^\circ)
  \]
Filtrations on $H^2(\mathbb{T}^2, C_a^o)$

- Hodge filtration:
  Let $\mathcal{E}^{-i}$ $(i = 0, 1, 2, \ldots)$ be the subspace of $\mathcal{R}_{F_a}$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.  

\[
\begin{array}{cccc}
0 & \subset & \mathcal{E}^0 = \mathbb{C}1 & \subset \mathcal{E}^{-1} & \subset \mathcal{E}^{-2} = \mathcal{R}_{F_a} \\
0 & \subset & F^2 & \subset F^1 & \subset F^0 = H^2(\mathbb{T}^2, C_a^o)
\end{array}
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\[
\begin{array}{cccc}
0 & \subset & \mathcal{I}_1 & \subset \mathcal{I}_2 & \subset \mathcal{I}_3 & \subset \mathcal{I}_4 = \mathcal{R}_{F_a} \\
0 & \subset & W_1 & \subset W_2 = W_3 & \subset W_4 = H^2(\mathbb{T}^2, C_a^o)
\end{array}
\]
Summary of MHS

\( C_a = \text{a compactification of } C_a^0, \text{ genus } C_a = 1. \)

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<thead>
<tr>
<th></th>
<th>( F^2 )</th>
<th>( F^1/F^2 )</th>
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<tbody>
<tr>
<td>( W_1 )</td>
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<td>( \mathbb{C}t_0 )</td>
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</table>
Gauss–Manin connection

- So far, the parameter $a$ of $C_a$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.

- The Gauss–Manin connection $\nabla$ corresponds to the differential operators on $\mathcal{R}_{F_a} \otimes \mathbb{C}(a)$ [Batyrev, Stienstra]:

\[ \nabla a_m =: \nabla a_m \iff D_a m := \partial a_m + t_0 t^m \quad (m \in \Delta) \]
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$$\nabla \partial_{a_m} =: \nabla a_m \Leftrightarrow D_{a_m} := \partial_{a_m} + t_0 t^m \quad (m \in \Delta)$$
GM connection

- How $\nabla_{a_m}$ acts on $H^2(\mathbb{T}^2, C_{a_0}^\circ) \cong \mathcal{R}_{F_a}$:

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- By the GM connection $\nabla$,
  - The Weight filtration is preserved: $\nabla_{a_m} W_k \subset W_k$
  - The Hodge filtration is changed by 1: $\nabla_{a_m} F^k \subset F^{k+1}$

(Griffiths transversality)
GM connection

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**GM connection**

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3 Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^\circ_a)$

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5 Holomorphic anomaly equation

6 Witten’s Geometric Quantization Approach

7 Appendix (Examples etc.)
Yukawa coupling

- In the case of $\mathcal{H}^3(X^\vee)$ of a Calabi–Yau threefold $X^\vee$, the Yukawa coupling is

$$
\int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega =: C_{ijk}.
$$

- In this definition, the polarization

$$
\mathcal{H}^3(X^\vee) \times \mathcal{H}^3(X^\vee) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{X^\vee} \alpha \wedge \beta
$$

is necessary.

- In the case of $\mathcal{H}^2(\mathbb{T}^2, C_\alpha^\circ)$, we note that

$$
\mathcal{W}_1 \mathcal{H}^2(\mathbb{T}^2, C_\alpha^\circ) = \mathcal{H}^1(C_\alpha)
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and use the polarization on $\mathcal{H}^1(C_\alpha)$ instead.
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Definition of Yukawa coupling

- Recall

\[
\begin{array}{c|ccc}
 & F^2 & F^1/F^2 & F^0/F^1 \\
\hline
H^1(C_a) = W_1 &  &  &  \\
W_2/W_1 &  &  &  \\
W_3/W_2 &  &  &  \\
W_4/W_3 & \mathbb{C}1 &  &  \\
\end{array}
\]

\[\nabla a_0 \rightarrow \mathbb{C}t_0 \rightarrow \mathbb{C}t_0^2\]

\[\nabla a_0 \rightarrow R^1_F/\mathbb{C}t_0\]

\[\omega = \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \in H^2(\mathbb{T}^2, C_a^0)\]

- Therefore \(\int_{C_a} \nabla^2 a_0 \omega \wedge \nabla a_0 \omega\) is well-defined.

- We define this as the Yukawa coupling \(\langle \partial a_0, \partial a_0; \partial a_0 \rangle\)
### Definition of Yukawa coupling

- **Recall**

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\[ \omega = \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} , 0 \right) \in H^2(\mathbb{T}^2, C_a^0) \]

- Therefore $\int_{\mathbb{C}a} \nabla_{a_0}^2 \omega \wedge \nabla_{a_0} \omega$ is well-defined.

- We define this as the Yukawa coupling $\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle$
Definition of Yukawa coupling

- Recall

\[ H^1(C_a) = W_1 \]

\[ F^2 \quad F^1/F^2 \quad F^0/F^1 \]

\[ W_2/W_1 \]

\[ W_3/W_2 \]

\[ W_4/W_3 \]

\[ \mathbb{C}1 \]

\[ \omega = \left( \frac{dt_1}{t_1} \land \frac{dt_2}{t_2}, 0 \right) \in H^2(\mathbb{T}^2, C_a^0) \]

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- We define this as the Yukawa coupling \[ \langle \partial_{a_0}, \partial_{a_0}, \partial_{a_0} \rangle \]
• The above pairing can be generalized to other vector fields as follows. Let
  • \( \mathbb{L} \): the base space of the family (space of the parameter \( a_m \)'s)
  • \( T^0 \mathbb{L} \): the subbundle of \( T \mathbb{L} \) spanned by \( \partial_{a_0} \)
  • The Yukawa coupling is a multilinear map:

\[
T \mathbb{L} \times T \mathbb{L} \times T^0 \mathbb{L} \rightarrow O_{\mathbb{L}},
\]

\[
\langle A, B; C \rangle := \int_{C_a} (\nabla_A \nabla_B \omega)' \wedge \nabla_C \omega
\]

• \( \nabla_C \omega \in F^1 \cap W_1 \) is a \((1, 0)\)-form on \( C_a \)
• \( \nabla_A \nabla_B \omega \) may be outside of \( W_1 \). But such a class can be written as

\[
\nabla_A \nabla_B \omega = \alpha_1 + \alpha_2 \quad \text{\((1, 0)\)-form on } C_a \text{\ with poles } + \text{\((0, 1)\)-form on } C_a \text{\ (without poles)}
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Remarks

- We can compute the Yukawa coupling by solving differential equations coming from A-hypergeometric system.

- Essentially, only \( \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle \) is relevant:

  \[
  \langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = f_{ij}(a) \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle,
  \]

  where

  \[
  \nabla_{a_i} \nabla_{a_j} \omega = t_0^2 t^{i+j} = f_{ij}(a) t_0^2 + (t_0\text{-degree} \leq 1).
  \]

- Ex.

\[
\Delta = \begin{array}{c}
  \text{\textcircled{}}
\end{array}
\]

\[
\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle = \frac{\text{const}}{a_0^3(1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})
\]

\[
\langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = \begin{cases} 
  \frac{9 a_0^2}{a_i a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i, j \neq 0) \\
  \frac{3 a_0}{a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i = 0)
\end{cases}
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(Same as the one obtained as the limit of the Yukawa coupling of the mirror of \( X \subset \mathbb{P}(1, 1, 1, 6, 9) \).)
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- We can compute the Yukawa coupling by solving differential equations coming from $A$-hypergeometric system.
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Outline

1. Motivation
2. Jacobian ring description of $H^2(\mathbb{T}^2, C_a^o)$
3. Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_a^o)$
4. Yukawa coupling
5. Holomorphic anomaly equation
6. Witten’s Geometric Quantization Approach
7. Appendix (Examples etc.)
Holomorphic Anomaly Eq.

- In the B-model of mirror symmetry, there is BCOV’s holomorphic anomaly equation. It is a system of differential equations for higher genus prepotentials $F_g (g \geq 1)$.

- Let $\mathcal{M}$ be the complex moduli space of a Calabi–Yau 3-fold $X^3$, and $z_1, \ldots, z_n$ be its local coordinates.

- Holomorphic anomaly eq. involves:
  - Kähler potential of $\mathcal{M}$: $K = -\log \sqrt{-1} \int_{X^3} \Omega \wedge \overline{\Omega}$
  - Kähler metric on $\mathcal{M}$: $G_{ij} = \partial_i \overline{\partial}_j K$
  - Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, T\mathcal{M}^\otimes 3)$

- Holomorphic anomaly equation:

$$\overline{\partial}_i F_g = \frac{1}{2} \sum_{j,k,j',k'} C_{ijk} e^{2K} G^{j'i'} G^{k'\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})$$
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Quotient family

- To formulate the holomorphic anomaly equation for local B-model, it is convenient to introduce the quotient family of our family of affine curves:
  - Consider the $\mathbb{T}^3$-action
    \[ \mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2). \]
  - $\mapsto$ the action on the parameter space $\mathbb{L}$ and the family
  - $\mapsto$ the quotient family
  - $\mathcal{M} := \mathbb{L}/\mathbb{T}^3$
  - (The above definition of the Yukawa coupling is also valid for the quotient family.)
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Special Kähler geometry

- It is necessary to modify the special Kähler geometry to the setting of local B-model.
  - \( Z_1, Z_2, \ldots, Z_{\dim \mathcal{M}}: \) local coordinates of \( \mathcal{M} \), \( \partial_i := \partial_{Z_i} \)
  - \( T^0 \mathcal{M} \subset T\mathcal{M} \): subbundle spanned by the image of \( a_0 \partial a_0 =: \theta_0 \)
  - Denote \( (\theta_i := z_i \partial_i, \partial_i := \partial_{z_i}) \)
    \[ Y_{i,j;0} := \langle \theta_i, \theta_j; \theta_0 \rangle \]
  - Hermitian metric on \( T^0 \mathcal{M} \):
    \[ (\theta_0, \theta_0) = \int_{\mathcal{C}_a} \nabla_{\theta_0} \omega \wedge \overline{\nabla_{\theta_0} \omega} =: G_{0\bar{0}} \]
  - \( Y_{ij;0} \) and \( G_{0\bar{0}} \) satisfy the relation (analogue of special geometry relation):
    \[ \overline{\partial_j} \frac{\theta_i G_{0\bar{0}}}{G_{0\bar{0}}} = -\frac{Y_{i0;0} \overline{Y}_{j0;0}}{G_{0\bar{0}}} \quad (\overline{\partial_j} = \overline{z} \partial_z) \]
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Special Kähler geometry

- It is necessary to modify the special Kähler geometry to the setting of local B-model.
- \( z_1, z_2, \ldots, z_{\dim M} \): local coordinates of \( M \), \( \partial_i := \partial_{z_i} \)
- \( T^0 M \subset T M \): subbundle spanned by the image of \( a_0 \partial a_0 =: \theta_0 \)

- Denote \( \theta_i := z_i \partial_i, \partial_i := \partial_{z_i} \)
  \[ Y_{i,j;0} := \langle \theta_i, \theta_j; \theta_0 \rangle \]

- Hermitian metric on \( T^0 M \):
  \[ (\theta_0, \theta_0) = \int_{C_a} \nabla_{\theta_0} \omega \wedge \overline{\nabla_{\theta_0} \omega} =: G_{0\bar{0}} \]

- \( Y_{i,j;0} \) and \( G_{0\bar{0}} \) satisfy the relation (analogue of special geometry relation):
  \[ \bar{\partial}_j \frac{\theta_i G_{0\bar{0}}}{G_{0\bar{0}}} = - \frac{Y_{i0;0} \bar{Y}_{j0;0}}{G_{0\bar{0}}} \quad (\bar{\partial}_j = \bar{z} \partial_{\bar{z}_j}) \]
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Holomorphic Anomaly Eq.

- Let $\tilde{C}_n^{(g)}$ be the “B-model topological string amplitude”

Set

$$
\begin{align*}
\tilde{C}_0^{(0)} &= \tilde{C}_1^{(0)} = \tilde{C}_2^{(0)} = 0, \quad \tilde{C}_3^{(0)} = Y_{00;0} \\
\tilde{C}_0^{(1)} &= 0 \\
\tilde{C}_n^{(g)} &= \left( \theta_0 - n\frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}} \right) \tilde{C}_n^{(g)}
\end{align*}
$$

(1)

- Holomorphic anomaly equation for local B-model is

$$
\begin{align*}
\bar{\theta}_j \tilde{C}_1^{(1)} &= -\frac{1}{2} \bar{\theta}_j \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}} \\
\bar{\theta}_j \tilde{C}_0^{(g)} &= \frac{Y_{j,0;0}}{2G_0^2} (\tilde{C}_2^{(g-1)} + \sum_{r=1}^{g-1} \tilde{C}_1^{(r)} \tilde{C}_1^{(g-r)}) \quad (g \geq 2)
\end{align*}
$$

(2)

$$
(\bar{\theta}_j = \bar{Z} \partial_{Z_j})
$$
Holomorphic Anomaly Eq.

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\tilde{C}^{(g)}_{n+1} &= \left( \theta_0 - n \frac{\theta G_{00\bar{0}}}{G_{00}} \right) \tilde{C}^{(g)}_n
\end{align*}$$

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\end{align*}
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\( \bar{\theta}_j = \bar{z} \partial_{\bar{z}_j} \)
Remarks

- This holo. anomaly eq. is consistent with the following observations made previously by several authors [Klemm–Zaslow, Hosono, Haghhiat–Klemm–Rauch, Aganagic–Bouchard–Klemm, Alim–Länger–Mayr].
  - Contains no Kähler potential;
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Solutions

- \( \tilde{C}_n^{(g)} \) is a polynomial of degree \( 3g - 3 + n \) in \( \mathbb{C}(z)[A] \), with
  \[
  A = \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}}. 
  \]

- For \((g, n) = (1, 1), (2, 0), \)
  \[
  \tilde{C}_1^{(1)} = \frac{1}{2} A + f_1^1(z) 
  \]
  \[
  \tilde{C}_0^{(2)} = -\frac{1}{2 Y_{00;0}} \left[ \frac{5}{12} A^3 - \left( \frac{\theta_0 Y_{00;0}}{4 Y_{00;0}} \right) A^2 
  + \left( -\frac{\kappa(z)}{2} + \theta_0 f_1^1(z) + (f_1^1(z))^2 \right) A \right] + f_2(z) 
  \]

- where
  - \( \kappa(z) \) is determined from special geometry.
  - \( f_1^1(z), f_0^2(z) \) are holomorphic (meromorphic) functions which cannot be determined from the holomorphic anomaly eq. (holomorphic ambiguities).
Local B-model and MHS
Yukiko Konishi

Motivation
Jacobian ring description of $H^2(\mathbb{T}^2, C_a^0)$
Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_a^0)$
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Outline

1 Motivation

2 Jacobian ring description of $H^2(\mathbb{T}^2, \mathcal{C}_a^\circ)$

3 Mixed Hodge Structure of $H^2(\mathbb{T}^2, \mathcal{C}_a^\circ)$

4 Yukawa coupling

5 Holomorphic anomaly equation

6 Witten’s Geometric Quantization Approach

7 Appendix (Examples etc.)
Witten’s Approach

- Witten gave a heuristic interpretation of BCOV’s holomorphic anomaly equation in the framework of geometric quantization (’93):
  - Holomorphic anomaly eq. $\iff$ A proj. flat connection on infinite dim. vector bundle $\mathcal{H} \rightarrow \mathcal{M}$

- His approach can be fit into the setting of local B-model and our holomorphic anomaly equation.
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Applying Geometric Quantization

- Consider the symplectic vector space $W = H^1(C_z, \mathbb{R})$ with the symplectic form given by the cup product.
- For each parameter $z \in \mathcal{M}$, let $x, \bar{x}$ be the complex coordinates of $W_\mathbb{C}$ associated to the basis $\phi = \nabla_{\theta_0} \omega \ (\text{holo.1-form}), \bar{\phi}$
- Let $L = W \times \mathbb{C}$ be the complex line bundle with the connection $D = \delta + \frac{1}{2} G_{0\bar{0}}(xd\bar{x} - \bar{x}dx)$.
  (Remark: $x, \bar{x}$ and $D$ depend on $z \in \mathcal{M}$ !)
- Consider the space of (square integrable) polarized sections $\mathcal{H}_z = \{ \Phi \in \Gamma(W, L) \mid D_{\partial_{\bar{x}}} \Phi = 0 \}$
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HAE as flat connection

- Witten’s idea is to interpret holomorphic anomaly eq. as a connection on the infinite dimensional vector bundle $\mathcal{H} \rightarrow \mathcal{M}$.

- In the case of local B-model, consider the connection $d$

\[ d(\theta_j) = \theta_j \]

\[ d(\tilde{\theta}_j) = \tilde{\theta}_j - \frac{Y_{j0;0}}{2G_{0\bar{0}}^2} \left( \delta_x - \frac{G_{0\bar{0}}}{2} \bar{x} \right)^2 \]

- Then $d$ is a projectively flat connection on $\mathcal{H}$.

- AND

\[ \exp \left[ \sum_{n,g} \frac{\lambda^{2g-2+n}}{n!} \tilde{C}^{(g)}_n x^n \right] \]

\[ \times e^{-\frac{G_{0\bar{0}}}{2} x \bar{x}} \]

\{ $C^{(g)}_n$ \} satisfies hol. anomaly eq.
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\{ C_n^{(g)} \} \text{ satisfies hol. anomaly eq.} \iff \exp \left[ \sum_{n,g} \frac{\lambda^{2g-2+n}}{n!} \tilde{C}_n^{(g)} x^n \right] \times e^{-\frac{G_{0\bar{0}}}{2} x \bar{x}}
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Outline

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2. Jacobian ring description of $H^2(\mathbb{T}^2, C^\circ_\alpha)$

3. Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^\circ_\alpha)$

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7. Appendix (Examples etc.)
Reflexive Polyhedra

- A reflexive polyhedron $\Delta$ is a polyhedron satisfying:
  - it is a convex hull of integral points;
  - $0 \in \Delta$;
  - Distance between 0 and each codimension 1 face is 1.
- There are 16 2-dimensional reflexive polyhedra.
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\[ \Delta \sim \mathbb{P} \]

- Regard integral points of \( \Delta \) other than the origin as one dimensional cones in \( \mathbb{R}^2 \).
- Then they define a complete smooth 2-dimensional fan.
- This fan in turn defines a complete smooth toric surface \( \mathbb{P} \) whose anti-canonical divisor is nef.

**Example**

\[
\begin{align*}
\Delta &= \begin{array}{c}
\text{v}^1 \\
\end{array} \
\text{v}^2 \
\text{v}^3 \\
\text{v}^1 &\quad \text{v}^2 &\quad \text{v}^3 \\
1 &\quad 0 &\quad 1 \\
0 &\quad 1 &\quad -1 \\
\end{array} 
\rightarrow \quad \mathbb{P} &= \mathbb{P}^2 \\
\text{v}^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{v}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{v}^3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.
\end{align*}
\]
- Regard integral points of $\Delta$ other than the origin as one dimensional cones in $\mathbb{R}^2$.
- Then they define a complete smooth 2-dimensional fan.
- This fan in turn defines a complete smooth toric surface $\mathbb{P}$ whose anti-canonical divisor is nef.

**Example**

![Diagram](image)

$\Delta = \begin{array}{c} v_2 \\ v_1 \\ v_3 \end{array} \rightarrow \mathbb{P} = \mathbb{P}^2$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. 
• Regard integral points of $\Delta$ other than the origin as one dimensional cones in $\mathbb{R}^2$.
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**Example**

\[
\Delta = \begin{array}{c} \downarrow \\ \searrow \\ \swarrow \end{array} \rightarrow \quad \mathbb{P} = \mathbb{P}^2
\]

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v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.
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**Example**

$$
\Delta = \begin{array}{c}
\downarrow \\
V_1 \\
\downarrow \\
V_2 \\
\downarrow \\
V_3
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$$

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$$
Local Gromov–Witten invariants

- Genus $g$ local Gromov–Witten invariant $N_{g, \beta}(\mathbb{P})$ of degree $\beta \in H_2(\mathbb{P}, \mathbb{Z})$ is defined by

$$N_{g, \beta}(\mathbb{P}) = \int_{[\overline{M}_{g,0}(\mathbb{P}, \beta)]^{vir}} e(\pi_* ev^* K_{\mathbb{P}}).$$

- Here
  - $\overline{M}_{g,n}(\mathbb{P}, \beta)$ is the moduli stack of stable maps to $\mathbb{P}$ of genus $g$ and degree $\beta$,
  - $ev : \overline{M}_{g,1}(\mathbb{P}, \beta) \to \mathbb{P}$ is the evaluation map,
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  - $e$ denotes the Euler class.
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Back To LMS
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\[ \Delta \rightsquigarrow C_a^o \]

- Associate the Laurent monomial to an integral point in \( \Delta \):
  \[ (m_1, m_2) \longleftrightarrow t^m := t_1^{m_1} t_2^{m_2} \]

- Take the sum of these monomials with parameters \( a_m \):
  \[ F_a(t) := \sum_{m \in \Delta} a_m t^m \]

- Example:
  \[ \Delta = \begin{array}{c}
  \begin{array}{c}
  \hline
  \hline
  a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.
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- The zero set of \( F_a(t) \) is an affine curve in \( \mathbb{T}^2 \):
  \[ C_a^o = \{(t_1, t_2) \in \mathbb{T}^2 \mid F_a(t) = 0 \} . \]

- \( C_a^o = \) genus 1 complete curve \( C_a \) – points

- Remark: we must put the \( \Delta \)-regularity condition on \( a_m \) so that \( C_a^o \) and its compactification \( C_a \) are smooth.
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Why $H^2(\mathbb{T}^2, C^o_\alpha)$, Not $H^1(C^o_\alpha)$ ?

1. $H^2(\mathbb{T}^2, C^o_\alpha)$ has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

VHS on $H^3$
- Hodge filtration
0 $\subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$
  has dim $F^3 = 1$
- $H^3$ generated by holo. 3-form $\in F^3$
  and GM connection

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2. Period integrals of $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$ satisfy the A-hypergeometric system.

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Example of $\Delta(k)$

$\Delta(2)$

$\Delta(1) = \Delta$

$\Delta(0)$
Examples of $\mathcal{E}$

- Regard each integral point $m$ in $\Delta(k)$ as the Laurent monomial $t_0^k t^m$. 
Example of $\mathcal{I}$

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![Diagram with points labeled $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$]
Example of quotient family 1

- $\mathcal{M} \cong \mathbb{P}(1, 3) \setminus \{0, \frac{1}{27}\}$
- A local coordinate (around 0) is $z = \frac{a_1a_2a_3}{a_0}$.
- $C_a \sim \{(t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z}{t_1t_2} = 0\}$
- Yukawa coupling is

$$\langle \partial_z, \partial_z; \partial_z \rangle = \frac{\text{const}}{27z^3(1 + 27z)}.$$
Example of quotient family 1

\[ \triangle = \]

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Example of quotient family 2

\[ \Delta = \begin{array}{c}
\end{array} \]

- \( M \subset \mathbb{P}^2 \) (open subset)
- Local coordinates:
  \[ z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_0^2} \]
- \( C_a \sim \{ (t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z_1}{t_1} + \frac{z_2}{t_2} = 0 \} \)
- Yukawa coupling is
  \[ \langle \theta_1, \theta_1, \theta_0 \rangle = \frac{8z_1}{d(z_1, z_2)}, \quad \langle \theta_1, \theta_2, \theta_0 \rangle = \frac{(1 - 4z_1 - 4z_2)}{d(z_1, z_2)}, \]
  where \( d(z_1, z_2) = (1 - 4z_1 - 4z_2) - 64z_1 z_2 \) and
  \[ \theta_0 = -2z_1 \frac{\partial}{\partial z_1} - 2z_2 \frac{\partial}{\partial z_1} = a_0 \frac{\partial}{\partial a_0} \]
Example of quotient family 2

- $\Delta = \Diamond$
- $\mathcal{M} \subset \mathbb{P}^2$ (open subset)

Local coordinates:

\[
z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_0^2}
\]

- $C_a \ni \{(t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z_1}{t_1} + \frac{z_2}{t_2} = 0\}$

Yukawa coupling is

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Example of quotient family 2

- $\Delta = \diamondsuit$
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- Yukawa coupling is
  \[ \langle \theta_1, \theta_1, \theta_0 \rangle = \frac{8z_1}{d(z_1, z_2)}, \quad \langle \theta_1, \theta_2, \theta_0 \rangle = \frac{(1 - 4z_1 - 4z_2)}{d(z_1, z_2)}, \]
  where $d(z_1, z_2) = (1 - 4z_1 - 4z_2) - 64z_1 z_2$ and
  \[ \theta_0 = -2z_1 \frac{\partial}{\partial z_1} - 2z_2 \frac{\partial}{\partial z_1} = a_0 \frac{\partial}{\partial a_0} \]
Example of quotient family 2

- $\triangle = \blacklozenge$
- $\mathcal{M} \subset \mathbb{P}^2$ (open subset)
- Local coordinates:
  \[
  z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_0^2}
  \]
- $C_a \rightsquigarrow \{(t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z_1}{t_1} + \frac{z_2}{t_2} = 0\}$
- Yukawa coupling is
  \[
  \langle \theta_1, \theta_1, \theta_0 \rangle = \frac{8z_1}{d(z_1, z_2)}, \quad \langle \theta_1, \theta_2, \theta_0 \rangle = \frac{(1 - 4z_1 - 4z_2)}{d(z_1, z_2)},
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  \]
About definition of $\tilde{C}_n^g$

- At present, there is no clear mathematical definition for "B-model topological string amplitude" $\tilde{C}_n^{(g)}$ except for $(g, n) = (0, 3)$. To give a mathematical definition is a very important problem.
Example of $\kappa$

- $\kappa$ is defined by

$$\kappa(z) := \theta_0 A + A^2 - \frac{\theta_0 Y_{00;0}}{Y_{00;0}} A.$$ 

- $\Delta = \triangle$

$$\kappa(z) = \frac{-54z}{1 + 27z}$$

- $\Delta = \square$

$$\kappa = \frac{8(z_1 + z_2 - 6(z_1^2 + z_2^2) + 12z_1z_2)}{(1 - 4z_1 - 4z_2)^2 - 64z_1z_2}$$
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- $\Delta = \begin{array}{c}
\frac{\theta_0}{Y_{00;0}} A
\end{array}$

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Holomorphic Ambiguities

- For \(\Delta = \triangle\)

\[
f_1^{(1)}(z) = \frac{1 + 54z}{4(1 + 27z)} ,
\]

\[
f_0^{(2)}(z) = \frac{1}{(1 + 27z)^2} \left( \frac{3}{40}z + \frac{783}{80}z^2 + \frac{3645}{8}z^3 \right)
\]

- How these are obtained:

- It is expected that \(\tilde{C}_1^{(1)}, \tilde{C}_0^{(g)} (g \geq 2)\) should be equal to the A-model topological string amplitudes (i.e. the generating functions of local GW invariants at each genus) under the mirror map \(t = t(z)\) and the "holomorphic limit"

\[
G_{00} \longrightarrow \theta_0 t(z) .
\]

- \(f_1^{(1)}(z), f_0^{(g)}(z) (g \geq 2)\) should be determined so that \(\tilde{C}_1^{(1)}, \tilde{C}_0^2\) reproduce the right local GW invariants.
Holomorphic Ambiguities

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  \begin{array}{c}
  \text{\Large \text{\textbullet}} \\
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  \includegraphics{triangle}
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