Local B-model and Mixed Hodge Structure

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Remark

In this talk:

- All manifolds (varieties) are complex.
- All variables and parameters are complex.
- $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$ $n$-dimensional complex torus, not $(S^1)^n$. 
Outline

1 Motivation

2 Jacobian ring description of $H^2(T^2, C^o_a)$

3 Mixed Hodge Structure of $H^2(T^2, C^o_a)$

4 Yukawa coupling

5 Holomorphic anomaly equation

6 Witten’s Geometric Quantization Approach

7 Appendix (Examples etc.)
Local Mirror Symmetry

- Local mirror symmetry is a variant of (ordinary) mirror symmetry.
- It is derived from mirror symmetry of toric Calabi–Yau hypersurfaces by considering a certain limits in moduli spaces. (e.g. CY hypersurface \( \subset \hat{\mathbb{P}}(1, 1, 1, 6, 9) \rightarrow \mathbb{P}^2 \)) [Katz–Klemm–Zaslow (1997), Chiang–Klemm–Yau–Zaslow (1999)]
- (Classical, not homological) local mirror symmetry is summarized as follows.
Local Mirror Symmetry

A-hypergeometric system with $\beta = \vec{0}$

Solutions give:
- Mirror map
- a derivative of prepotential

PF equation for period integrals of "top element"

$\Delta$

2 dim reflexive polyhedron

compact toric surface
$\mathbb{P}$ s.t. $-K_\mathbb{P}$ nef
$g = 0$ local GW inv.
local A-model

a family of affine curves
$C_\alpha \subset \mathbb{T}^2$
VMHS on $H^2(\mathbb{T}^2, C_\alpha)$
local B-model

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Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_\alpha)$
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$$\left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$$

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- A family of affine curves
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  VMHS on $H^2(\mathbb{T}^2, C_a^{\circ})$
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Comparison with Mirror Symmetry

**A**

GW inv of \(X\)

VHS on \(H^3(X^\vee)\)

- holo. 3-form \(\Omega\)
- Yukawa coupling

\[ \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega \]

**B**

**MS**

**LMS**

local GW inv of \(\mathbb{P}\)

VMHS on \(H^2(\mathbb{T}^2, C_a)\)

- \(\omega := \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)\)

- ?? ⇐ This talk

Important because it is:

- a third derivative of prepotential;
- necessary for BCOV’s holomorphic anomaly eq.
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- **MS**
  - GW inv of $X$
  - VHS on $H^3(X^\vee)$
    - holo. 3-form $\Omega$
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Our aim

- In several examples of local B-model, the Yukawa couplings have been computed [Klemm–Zaslow, Jinzenji–Forbes, Aganagic–Bouchard–Klemm, Haghihat–Klemm–Rauch, Alim–Länge-Mayr, Brini–Tanzini]. However, there has been no direct definition.

- We gave a definition of local B-model Yukawa coupling using the results of Batyrev, Stienstra on the VMHS on $H^2(\mathbb{T}^2, C_a^\circ)$.

- We also proposed how to modify Bershadsky–Cecotti–Ooguri–Vafa’s holomorphic anomaly equation to the setting of local B-model.
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Decryption of $H^2(\mathbb{T}^2, C_a^\circ)$

- $H^2(\mathbb{T}^2, C_a^\circ)$ was studied by Batyrev (’93) and Stienstra (’97).
- $H^2(\mathbb{T}^2, C_a^\circ)$ has a Jacobian-ring like description. It is isomorphic to a (quotient) vector space $\mathcal{R}_{F_a}$, which is determined by the data of $\Delta$ and $F_a(t)$.
- The mixed Hodge structure on $H^2(\mathbb{T}^2, C_a^\circ)$ is given in terms of filtrations of $\mathcal{R}_{F_a}$.
- The variation of mixed Hodge structures on $H^2(\mathbb{T}^2, C_a^\circ)$ is also described in terms of $\mathcal{R}_{F_a} (\nabla a_m \iff \text{Derivation by } a_m \text{ on } \mathcal{R}_{F_a})$. 
“Jacobian Ring” $\mathcal{R}_{F_a}$

- $\Delta(k)$: the polyhedron obtained by enlarging $\Delta$ by $k$-times.  

- Consider the (infinite dim) vector space spanned by monomials $t_0^k t^m$ \text{“lying on } \Delta(k)$$:  

$$S^k_\Delta := \bigoplus_{m \in \Delta(k)} C t_0^k t^m \quad (t^m := t_1^{m_1} t_2^{m_2})$$

$$\mathcal{S}_\Delta := \bigoplus_{k \geq 0} S^k_\Delta, \quad \deg t_0^k t^m := k \quad \text{(a graded ring)}$$

- Define the differential operators on $\mathcal{S}_\Delta$: $(\theta_x := x \partial_x)$

$$\mathcal{D}_0(t_0^k t^m) = (k + t_0 F_a(t)) t_0^k t^m$$

$$\mathcal{D}_i(t_0^k t^m) = (m_i + t_0 \theta_{t_i} F_a(t)) t_0^k t^m \quad (i = 1, 2)$$

- The “Jacobian ring” $\mathcal{R}_{F_a}$ is the quotient vector space:

$$\mathcal{R}_{F_a} := \mathcal{S}_\Delta / \left( \sum_{i=0}^{2} \mathcal{D}_i \mathcal{S}_\Delta \right)$$
Examples of $\mathcal{R}_{F_a}$

- $\Delta = \triangle$
  \[ F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} \]

Relations $\mathcal{D}_i 1 = 0$ ($i = 0, 1, 2$) imply:

\[
\begin{align*}
t_0 t_1 &= -\frac{a_0}{3a_1} t_0, \\
t_0 t_2 &= -\frac{a_0}{3a_2} t_0, \\
t_0 &\quad t_1 t_2 = -\frac{a_0}{3a_3} t_0.
\end{align*}
\]

By similar calculation, $t_0^k t^m (k \geq 2)$ is equal to

\[ \text{const.} t_0^2 + \text{term of } t_0\text{-degree 1} . \]

\[ \therefore \mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \]

- $\Delta = \square$

\[ \mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0 t_1 \oplus \mathbb{C}t_0^2 \]
\[ \mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C^\circ_a) \]

- For a (general) 2-dim reflexive polyhedron \( \Delta \),
  \[
  \mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus R^1_{F_a} \oplus \mathbb{C}t_0^2 \quad (\text{dim} = \#\Delta \cap \mathbb{Z}^2 - 1)
  \]
  \[
  R^1_{F_a} := S^1_\Delta / \mathbb{C}t_0 F_a \oplus \mathbb{C}t_1 \theta_{t_1} F_a \oplus \mathbb{C} \theta_{t_2} F_a \quad (\text{degree 1-part})
  \]

- Note that there is an exact sequence
  \[
  0 \longrightarrow PH^1(C^\circ_a) \longrightarrow H^2(\mathbb{T}^2, C^\circ_a) \longrightarrow H^2(\mathbb{T}^2) \longrightarrow 0
  \]
  \[
  (PH^1(C^\circ_a) := H^1(C^\circ_a)/H^1(\mathbb{T}^2))
  \]

- An isomorphism \( \mathcal{R}_{F_a} \overset{\sim}{\longrightarrow} H^2(\mathbb{T}^2, C^\circ_a) \) [Batyrev, Stienstra] is given by:
  \[
  \mathbb{C}1 \leftrightarrow H^2(\mathbb{T}^2) : 1 \leftrightarrow \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)
  \]
  \[
  R^1_{F_a} \oplus \mathbb{C}t_0^2 \leftrightarrow PH^1(C^\circ_a) : t_0^k t^m \leftrightarrow \left( 0, \text{Res}_{F_a=0} \frac{(k - 1)!t^m}{(-1)^{k-1}F_a^k} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \right)
  \]
Examples of $\mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C_a^\circ)$

- $C_a :=$ compactification of $C_a^\circ$, genus $C_a = 1$.

- $\Delta = \begin{array}{c}
\begin{array}{c}
\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \\
on \mathbb{T}^2
\end{array}
\end{array}$

- $\mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2$

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- $\mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0 t_1 \oplus \mathbb{C}t_0^2$

- (1, 0)-form on $C_a$

- (0, 1)-form on $C_a$

- (1, 0)-form on $C_a$ with poles

- (0, 1)-form on $C_a$
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What’s Mixed Hodge Structure?

- $H^k(V)$ of a smooth projective variety $V$ has the canonical Hodge structure of weight $k$:

$$H^k(V) = \bigoplus_{p+q=k} H^{p,q}(V)$$

(Hodge decomposition)

- To generalize this to $H^k(U)$ of an open variety $U$, it is necessary to consider the mixed Hodge structure.

- Roughly speaking, the mixed Hodge structure is the direct sum of Hodge structures of different weights:

$$\bigoplus_{p+q=l} H^{p,q}$$
Definition of MHS

- Mixed Hodge structure of weight \( k \) consists of:
  - free abelian group \( H_\mathbb{Z} \),
  - the weight filtration \( W_* \) on \( H_\mathbb{Z} \) (increasing filtration),
  - the Hodge filtration \( F^* \) on \( H_\mathbb{C} \) (decreasing filtration),

- such that the induced Hodge filtration on \( W_I/W_{I-1} \) has a Hodge structure of weight \( I + k \).

\[
H^{p,k+I-p} := \frac{F^p W_I/W_{I-1}}{F^{p+1} W_I/W_{I-1}} \text{ satisfy } H^{p,q} = \overline{H}^{q,p}.
\]
MHS for an open variety

- If $U = V - D$ where $V$ is a smooth projective variety and $D$ is a simple normal crossing divisor, then $H^k(U)$ has a canonical mixed Hodge structure.

- Hodge filtration $F^\bullet$ is induced from the filtration on $\Omega^\bullet_V(\log D)$

$$F^p \Omega^\bullet_V(\log D) = \Omega^{>p}_V(\log D)$$

- Weight filtration is induced from the filtration

$$W_l \Omega^\bullet_V(\log D) = \wedge^l \Omega^1_V(\log D) \wedge \Omega^{-l}_V$$

Roughly speaking, $W_{k+1} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

- For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

$$\ldots \to H^k(U_1) \to H^{k+1}(U_2, U_1) \to H^{k+1}(U_2) \to \ldots$$

is a long exact sequence of MHS’s.
Filtrations on $H^2(\mathbb{T}^2, C_a^\circ)$

- **Hodge filtration:**
  Let $\mathcal{E}^{-i} (i = 0, 1, 2, \ldots)$ be the subspace of $\mathcal{R}_{F_a}$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.

  $\begin{align*}
  0 & \subset \mathcal{E}^0 = \mathbb{C}1 \\ & \subset \mathcal{E}^{-1} \\ & \subset \mathcal{E}^{-2} = \mathcal{R}_{F_a}
  \end{align*}$

- **Weight filtration:**
  Let $\mathcal{I}_j (1 \leq j \leq 3) \subset \mathcal{R}_{F_a}$ be spanned by the images of $t_0^k t^m$'s such that $k \geq 1$, $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $\mathcal{I}_4 := \mathcal{R}_{F_a}$.

  $\begin{align*}
  0 & \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \quad \subset \mathcal{I}_4 = \mathcal{R}_{F_a}
  \end{align*}$
Summary of MHS

\((C_a = \text{a compactification of } C_a^0, \text{ genus } C_a = 1)\)

\[
\begin{array}{c|ccc}
 & F^2 & F^1/F^2 & F^0/F^1 \\
\hline
W_1 & \mathcal{H}^1(C_a) = \begin{pmatrix} \mathbb{C}t_0 \\ \mathbb{C}t_0^2 \end{pmatrix} & (1, 0)\text{-form on } C_a & (0, 1)\text{-form on } C_a \\
W_2/W_1 & R_F^1/\mathbb{C}t_0 & (1, 0)\text{-form on } C_a \text{ with poles at } C_a - C_a^0 \\
W_3/W_2 & \mathbb{C}1 & \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \text{ on } \mathbb{T}^2 \\
W_4/W_3 & \\
\end{array}
\]
Gauss–Manin connection

- So far, the parameter $a$ of $C_a$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.

- The Gauss–Manin connection $\nabla$ corresponds to the differential operators on $\mathcal{R}_{F_a} \otimes \mathbb{C}(a)$ [Batyrev, Stienstra]:

$$\nabla \partial_{am} =: \nabla a_m \iff D_{am} := \partial_{am} + t_0 t^m \quad (m \in \Delta)$$
GM connection

- How $\nabla_{a_m}$ acts on $H^2(\mathbb{T}^2, C^\circ_a) \cong \mathcal{R}_{F_a}$:

<table>
<thead>
<tr>
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<th>$F^2$</th>
<th>$F^1/F^2$</th>
<th>$F^0/F^1$</th>
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<tr>
<td>$W_1$</td>
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- By the GM connection $\nabla$,
  - The Weight filtration is preserved: $\nabla_{a_m} W_k \subset W_k$
  - The Hodge filtration is changed by 1: $\nabla_{a_m} F^k \subset F^{k+1}$
    (Griffiths transversality)
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Yukawa coupling

- In the case of $H^3(X^\vee)$ of a Calabi–Yau threefold $X^\vee$, the Yukawa coupling is

\[ \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega =: C_{ijk} . \]

- In this definition, the polarization

\[ H^3(X^\vee) \times H^3(X^\vee) \to \mathbb{C} , \quad (\alpha, \beta) \mapsto \int_{X^\vee} \alpha \wedge \beta \]

is necessary.

- In the case of $H^2(\mathbb{T}^2, C_a^\circ)$, we note that

\[ W_1 H^2(\mathbb{T}^2, C_a^\circ) = H^1(C_a) \]

and use the polarization on $H^1(C_a)$ instead.
Definition of Yukawa coupling

- Recall

<table>
<thead>
<tr>
<th>$H^1(C_a) = W_1$</th>
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<td>$\mathbb{C}1$</td>
<td>$\nabla a_0$</td>
<td>$\nabla a_0$</td>
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</table>

$\omega = \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \in H^2(\mathbb{T}^2, C_a^0)$

- Therefore $\int_{C_a} \nabla^2_{a_0} \omega \wedge \nabla a_0 \omega$ is well-defined.

- We define this as the Yukawa coupling $\langle \partial_{a_0}, \partial_{a_0} ; \partial_{a_0} \rangle$
• The above pairing can be generalized to other vector fields as follows. Let
  • \( \mathbb{L} \): the base space of the family (space of the parameter \( a_m \)'s)
  • \( T^0 \mathbb{L} \): the subbundle of \( T \mathbb{L} \) spanned by \( \partial_{a_0} \)

• The Yukawa coupling is a multilinear map:

\[
T \mathbb{L} \times T \mathbb{L} \times T^0 \mathbb{L} \to \mathcal{O}_\mathbb{L},
\]

\[
\langle A, B; C \rangle := \int_{C_a} (\nabla_A \nabla_B \omega)' \wedge \nabla_C \omega
\]

• \( \nabla_C \omega \in F^1 \cap W_1 \) is a \((1, 0)\)-form on \( C_a \)
• \( \nabla_A \nabla_B \omega \) may be outside of \( W_1 \). But such a class can be written as

\[
\nabla_A \nabla_B \omega = \alpha_1 \quad \text{\((1, 0)\)-form on } C_a \text{ with poles}
\]

\[
+ \alpha_2 \quad \text{\((0, 1)\)-form on } C_a \text{ (without poles)}
\]

So set \( (\nabla_A \nabla_B \omega)' := \alpha_2 \).
Remarks

- We can compute the Yukawa coupling by solving differential equations coming from A-hypergeometric system.

- Essentially, only $\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle$ is relevant:

\[ \langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = f_{ij}(a) \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle, \]

where $\nabla_{a_i} \nabla_{a_j} \omega = t_0^2 t^{i+j} = f_{ij}(a) t_0^2 + (t_0\text{-degree} \leq 1)$. 

- Ex. \[ \Delta = \begin{array}{c} \end{array} \]

\[ \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle = \frac{\text{const}}{a_0^3 (1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3}) \]

\[ \langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = \begin{cases} 
\frac{9 a_0^2}{a_i a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i, j \neq 0) \\
\frac{3 a_0}{a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i = 0) 
\end{cases} \]

(Same as the one obtained as the limit of the Yukawa coupling of the mirror of $X \subset \hat{\mathbb{P}}(1, 1, 1, 6, 9)$.)
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Holomorphic Anomaly Eq.

- In the B-model of mirror symmetry, there is BCOV’s holomorphic anomaly equation. It is a system of differential equations for higher genus prepotentials $F_g (g \geq 1)$.

- Let $\mathcal{M}$ be the complex moduli space of a Calabi–Yau 3-fold $X^\vee$, and $z_1, \ldots, z_n$ be its local coordinates.

- Holomorphic anomaly eq. involves:
  - Kähler potential of $\mathcal{M}$: $K = - \log \sqrt{-1} \int_{X^\vee} \Omega \wedge \bar{\Omega}$
  - Kähler metric on $\mathcal{M}$: $G_{ij} = \partial_i \bar{\partial}_j K$
  - Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes^3)$

- Holomorphic anomaly equation:

$$\bar{\partial}_i F_g = \frac{1}{2} \sum_{j,k,j,k} \bar{C}_{ijk} e^{2K} G^{j\bar{j}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})$$
Quotient family

- To formulate the holomorphic anomaly equation for local B-model, it is convenient to introduce the quotient family of our family of affine curves:

- Consider the $\mathbb{T}^3$-action

  \[ \mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2). \]

- the action on the parameter space $\mathbb{L}$ and the family
- the quotient family
- $\mathcal{M} := \mathbb{L}/\mathbb{T}^3$
- (The above definition of the Yukawa coupling is also valid for the quotient family.)

Example 1  Example 2
Special Kähler geometry

- It is necessary to modify the special Kähler geometry to the setting of local B-model.
- \( z_1, z_2, \ldots, z_{\dim \mathcal{M}} \): local coordinates of \( \mathcal{M} \), \( \partial_i := \partial z_i \)
- \( \mathcal{T}^0 \mathcal{M} \subset \mathcal{T} \mathcal{M} \): subbundle spanned by the image of
  \[
  a_0 \partial_{a_0} =: \theta_0
  \]
- Denote \( (\theta_i := z_i \partial_i, \partial_i := \partial z_i) \)
  \[
  Y_{i,j;0} := \langle \theta_i, \theta_j; \theta_0 \rangle
  \]
- Hermitian metric on \( \mathcal{T}^0 \mathcal{M} \):
  \[
  (\theta_0, \theta_0) = \int_{C_a} \nabla_{\theta_0} \omega \wedge \overline{\nabla_{\theta_0} \omega} =: G_{0\bar{0}}
  \]
- \( Y_{ij;0} \) and \( G_{0\bar{0}} \) satisfy the relation (analogue of special geometry relation):
  \[
  \overline{\theta_j} \frac{\theta_i G_{0\bar{0}}}{G_{0\bar{0}}} = - \frac{Y_{i0;0} \overline{Y}_{j0;0}}{G_{0\bar{0}}} \quad (\overline{\theta_j} = \overline{z} \partial_{\overline{z}_j})
  \]
Holomorphic Anomaly Eq.

- Let $\tilde{C}^{(g)}_n$ be the “B-model topological string amplitude”

- Set

$$
\begin{align*}
\tilde{C}^{(0)}_0 &= \tilde{C}^{(0)}_1 = \tilde{C}^{(0)}_2 = 0, \quad \tilde{C}^{(0)}_3 = Y_{00;0} \\
\tilde{C}^{(1)}_0 &= 0 \\
\tilde{C}^{(g)}_{n+1} &= \left( \theta_0 - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}} \right) \tilde{C}^{(g)}_n
\end{align*}
$$

(1)

- Holomorphic anomaly equation for local B-model is

$$
\begin{align*}
\bar{\theta}_j \tilde{C}^{(1)}_1 &= -\frac{1}{2} \bar{\theta}_j \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}} \\
\bar{\theta}_j \tilde{C}^{(g)}_0 &= \frac{Y_{j,0;0}}{2G_{0\bar{0}}^2} (\tilde{C}^{(g-1)}_2 + \sum_{r=1}^{g-1} \tilde{C}_1^{(r)} \tilde{C}_1^{(g-r)}) \quad (g \geq 2)
\end{align*}
$$

(2)

$(\bar{\theta}_j = \bar{Z} \partial_{Z_j})$
Remarks

- This holo. anomaly eq. is consistent with the following observations made previously by several authors [Klemm–Zaslow, Hosono, Haghhiat–Klemm–Rauch, Aganagic–Bouchard–Klemm, Alim–Länge–Mayr].
  - Contains no Kähler potential;
  - Only the one dimensional subbundle $T^0 \mathcal{M}$ of $TM$ matters. Similar to the one-parameter model.
- Holomorphic anomaly equation can be solved by using the Feynman diagrams (with only one propagator) and also by Yamaguchi–Yau’s method.
Solutions

- $\tilde{C}^{(g)}_n$ is a polynomial of degree $3g - 3 + n$ in $\mathbb{C}(z)[A]$, with
  \[ A = \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}}. \]

- For $(g, n) = (1, 1), (2, 0)$,
  \[ \tilde{C}_1^{(1)} = \frac{1}{2} A + f_1^1(z) \]
  \[ \tilde{C}_0^{(2)} = -\frac{1}{2 Y_{00;0}} \left[ \frac{5}{12} A^3 - \left( \frac{\theta_0 Y_{00;0}}{4 Y_{00;0}} \right) A^2 \right. \]
  \[ + \left. \left( -\frac{\kappa(z)}{2} + \theta_0 f_1^1(z) + (f_1^1(z))^2 \right) A \right] + f_2(z) \]

- where
  - $\kappa(z)$ is determined from special geometry.
  - $f_1^1(z), f_0^2(z)$ are holomorphic (meromorphic) functions which cannot be determined from the holomorphic anomaly eq. (holomorphic ambiguities)
Witten’s Approach

- Witten gave a heuristic interpretation of BCOV’s holomorphic anomaly equation in the framework of geometric quantization (’93):
  
  \[
  \text{Holomorphic anomaly eq.} \quad \iff \quad \text{A proj. flat connection on infinite dim. vector bundle } \mathcal{H} \to \mathcal{M}
  \]

- His approach can be fit into the setting of local B-model and our holomorphic anomaly equation.
Applying Geometric Quantization

- Consider the symplectic vector space $\mathcal{W} = H^1(C_z, \mathbb{R})$ with the symplectic form given by the cup product.

- For each parameter $z \in \mathcal{M}$, let $x, \bar{x}$ be the complex coordinates of $\mathcal{W}_\mathbb{C}$ associated to the basis

$$\phi = \nabla_\theta \omega \quad \text{(holo.1-form),} \quad \bar{\phi}$$

- Let $L = \mathcal{W} \times \mathbb{C}$ be the complex line bundle with the connection

$$D = \delta + \frac{1}{2} G_{0\bar{0}}(xd\bar{x} - \bar{x}dx) .$$

(Remark: $x, \bar{x}$ and $D$ depend on $z \in \mathcal{M}$ !)

- Consider the space of (square integrable) polarized sections

$$\mathcal{H}_z = \{ \Phi \in \Gamma(\mathcal{W}, L) \mid D_{\partial \bar{x}} \Phi = 0 \}$$
HAE as flat connection

- Witten's idea is to interpret holomorphic anomaly eq. as a connection on the infinite dimensional vector bundle $\mathcal{H} \to \mathcal{M}$.
- In the case of local B-model, consider the connection $d$
  \[ d(\theta_j) = \theta_j \]
  \[ d(\bar{\theta}_j) = \bar{\theta}_j - \frac{\bar{Y}_{j0;0}}{2G_{00}^2} \left( \delta x - \frac{G_{00}}{2} \bar{x} \right)^2 \]
- Then $d$ is a projectively flat connection on $\mathcal{H}$!
- AND

\[
\{ C_n^{(g)} \} \quad \text{satisfies hol. anomaly eq.} \quad \leftrightarrow \quad \exp \left[ \sum_{n,g} \frac{\lambda^{2g-2+n}}{n!} \tilde{C}_n^{(g)} x^n \right] 
\times e^{-\frac{G_{00}}{2} x \bar{x}} \quad \text{is a flat section}
\]
Outline

1 Motivation
2 Jacobian ring description of $H^2(\mathbb{T}^2, C^\circ_a)$
3 Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^\circ_a)$
4 Yukawa coupling
5 Holomorphic anomaly equation
6 Witten’s Geometric Quantization Approach
7 Appendix (Examples etc.)
Reflexive Polyhedra

- A reflexive polyhedron $\Delta$ is a polyhedron satisfying:
  - it is a convex hull of integral points;
  - $0 \in \Delta$;
  - Distance between 0 and each codimension 1 face is 1.
- There are 16 2-dimensional reflexive polyhedra.
• Regard integral points of $\Delta$ other than the origin as one dimensional cones in $\mathbb{R}^2$.
• Then they define a complete smooth 2-dimensional fan.
• This fan in turn defines a complete smooth toric surface $\mathbb{P}$ whose anti-canonical divisor is nef.

**Example**

\[ \Delta = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \quad \rightarrow \quad \mathbb{P} = \mathbb{P}^2 \]

\[ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \]
Local Gromov–Witten invariants

- Genus $g$ local Gromov–Witten invariant $N_{g,\beta}(\mathbb{P})$ of degree $\beta \in H_2(\mathbb{P}, \mathbb{Z})$ is defined by

$$N_{g,\beta}(\mathbb{P}) = \int_{[\overline{M}_{g,0}(\mathbb{P},\beta)]^{\text{vir}}} e(\pi_*ev^*K_{\mathbb{P}}).$$

- Here
  - $\overline{M}_{g,n}(\mathbb{P},\beta)$ is the moduli stack of stable maps to $\mathbb{P}$ of genus $g$ and degree $\beta$,
  - $ev : \overline{M}_{g,1}(\mathbb{P},\beta) \to \mathbb{P}$ is the evaluation map,
  - $\pi : \overline{M}_{g,1}(\mathbb{P},\beta) \to \overline{M}_{g,0}(\mathbb{P},\beta)$ is the map forgetting the marked point,
  - $e$ denotes the Euler class.
- **Remark.** This is defined for $\beta$ such that $\beta.K_{\mathbb{P}} \neq 0$. 

\[ \Delta \rightsquigarrow \mathcal{C}_a^o \]

- Associate the Laurent monomial to an integral point in \( \Delta \):
  \[
  (m_1, m_2) \longleftrightarrow t^m := t_1^{m_1} t_2^{m_2}
  \]

- Take the sum of these monomials with parameters \( a_m \):
  \[
  F_a(t) := \sum_{m \in \Delta} a_m t^m
  \]

- Example:
  \[
  \Delta = \begin{array}{c}
  \end{array}
  \]
  \[
  F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.
  \]

- The zero set of \( F_a(t) \) is an affine curve in \( \mathbb{T}^2 \):
  \[
  \mathcal{C}_a^o = \{(t_1, t_2) \in \mathbb{T}^2 \mid F_a(t) = 0\}.
  \]

- \( \mathcal{C}_a^o = \) genus 1 complete curve \( \mathcal{C}_a \) – points

- Remark: we must put the \( \Delta \)-regularity condition on \( a_m \) so that \( \mathcal{C}_a^o \) and its compactification \( \mathcal{C}_a \) are smooth.
Why $H^2(\mathbb{T}^2, C^o_\alpha)$, Not $H^1(C^o_\alpha)$?

1. $H^2(\mathbb{T}^2, C^o_\alpha)$ has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

   - **VHS on $H^3$**
     - Hodge filtration
     - $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$
     - has dim $F^3 = 1$
     - $H^3$ generated by holo. 3-form $\in F^3$
     - and GM connection

   - **VMHS on $H^2(\mathbb{T}^2, C^o_\alpha)$**
     - Hodge filtration
     - $0 \subset F^2 \subset F^1 \subset F^0 = H^2$
     - has dim $F^2 = 1$
     - $H^2$ generated by
     - $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in F^2$
     - and GM connection

2. Period integrals of $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$ satisfy the A-hypergeometric system.

3. $H^1(C^o_\alpha)$ does not have these properties!
Example of $\Delta(k)$

$\Delta(2)$

$\Delta(1) = \Delta$

$\Delta(0)$
Examples of $\mathcal{E}$

- Regard each integral point $m$ in $\Delta(k)$ as the Laurent monomial $t_0^k t^m$. 

\begin{itemize}
    \item $\mathcal{E}^0$
    \item $\mathcal{E}^{-1}$
    \item $\mathcal{E}^{-2}$
\end{itemize}
Example of $\mathcal{I}$

- Regard each integral point $m$ in $\Delta(k)$ as the Laurent monomial $t_0^k t^m$.
Example of quotient family 1

\[ \Delta = \]

- \( \mathcal{M} \cong \mathbb{P}(1, 3) \setminus \{0, \frac{1}{27}\} \)
- A local coordinate (around 0) is \( z = \frac{a_1 a_2 a_3}{a_0^3} \).
- \( C_a \sim \{(t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z}{t_1 t_2} = 0\} \)
- Yukawa coupling is

\[
\langle \partial_z, \partial_z; \partial_z \rangle = \frac{\text{const}}{27z^3(1 + 27z)}.
\]
Example of quotient family 2

1. \( \Delta = \) 
2. \( \mathcal{M} \subset \mathbb{P}^2 \) (open subset)
3. Local coordinates:
   \[ z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_0^2} \]
4. \( \mathcal{C}_a \cong \{(t_1, t_2) \mid 1 + t_1 + t_2 + \frac{z_1}{t_1} + \frac{z_2}{t_2} = 0\} \)
5. Yukawa coupling is
   \[ \langle \theta_1, \theta_1, \theta_0 \rangle = \frac{8z_1}{d(z_1, z_2)}, \quad \langle \theta_1, \theta_2, \theta_0 \rangle = \frac{(1 - 4z_1 - 4z_2)}{d(z_1, z_2)}, \]
   where \( d(z_1, z_2) = (1 - 4z_1 - 4z_2) - 64z_1z_2 \) and
   \[ \theta_0 = -2z_1 \frac{\partial}{\partial z_1} - 2z_2 \frac{\partial}{\partial z_1} = a_0 \frac{\partial}{\partial a_0} \]
About definition of $\tilde{C}_n^g$

- At present, there is no clear mathematical definition for “B-model topological string amplitude” $\tilde{C}_n^{(g)}$ except for $(g, n) = (0, 3)$. To give a mathematical definition is a very important problem.
Example of $\kappa$

- $\kappa$ is defined by

$$\kappa(z) := \theta_0 A + A^2 - \frac{\theta_0 Y_{00;0}}{Y_{00;0}} A.$$ 

- $\Delta = \begin{array}{c}
\cdot
\end{array}$

$$\kappa(z) = \frac{-54z}{1 + 27z}$$

- $\Delta = \begin{array}{c}
\cdot
\end{array}$

$$\kappa = \frac{8(z_1 + z_2 - 6(z_1^2 + z_2^2) + 12z_1z_2)}{(1 - 4z_1 - 4z_2)^2 - 64z_1z_2}$$
Holomorphic Ambiguities

• For \( \Delta = \begin{array}{c}
  \hline
  & & \\
  & & \\
  & & \\
  \end{array} \)

\[
f_1^{(1)}(z) = \frac{1 + 54z}{4(1 + 27z)},
\]

\[
f_0^{(2)}(z) = \frac{1}{(1 + 27z)^2} \left( \frac{3}{40} z + \frac{783}{80} z^2 + \frac{3645}{8} z^3 \right)
\]

• How these are obtained:

• It is expected that \( \tilde{C}_1^{(1)}, \tilde{C}_0^{(g)} \) \((g \geq 2)\) should be equal to the A-model topological string amplitudes (i.e. the generating functions of local GW invariants at each genus) under the mirror map \( t = t(z) \) and the “holomorphic limit”

\[
G_{00} \longrightarrow \theta_0 t(z).
\]

• \( f_1^{(1)}(z), f_0^{(g)}(z) \) \((g \geq 2)\) should be determined so that \( \tilde{C}_1^{(1)}, \tilde{C}_0^2 \) reproduce the right local GW invariants.