Local B-model and Mixed Hodge Structure

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Joint work with Satoshi Minabe (Tokyo Denki Univ.)
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Outline

1. Motivation
2. Jacobian ring description of $H^2(\mathbb{T}^2, \mathbb{C}_a)$
3. Mixed Hodge Structure of $H^2(\mathbb{T}^2, \mathbb{C}_a)$
4. Yukawa coupling
5. Holomorphic anomaly equation
6. Witten's Geometric Quantization Approach
7. Appendix (Local GW invariant)
Local Mirror Symmetry

- Local mirror symmetry (LMS) is a variant of mirror symmetry.
- It is derived from mirror symmetry of toric Calabi–Yau hypersurfaces by considering a certain limits in moduli spaces. (e.g. CY hypersurface $\subset \hat{\mathbb{P}}(1, 1, 1, 6, 9) \sim \mathbb{P}^2$) [Katz–Klemm–Zaslow (1997), Chiang–Klemm–Yau–Zaslow (1999)]

- LMS can be formulated without referring to the above limiting procedure.
- LMS is summarized as follows.
Reflexive Polyhedra

- A reflexive polyhedron \( \Delta \) is a polyhedron satisfying:
  - it is a convex hull of integral points;
  - \( 0 \in \Delta \);
  - Distance between 0 and each codimension 1 face is 1.

- There are 16 2-dimensional reflexive polyhedra [Batyrev]. To each, one example of LMS is associated.
Local Mirror Symmetry

A-hypergeometric system with $\beta = \vec{0}$

Solutions give:
- Mirror map
- a derivative of prepotential

$\Delta$
2 dim reflexive polyhedron

PF equation for period integrals of "top element"
$\left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$

compact toric surface
$\mathbb{P}$ s.t. $-K_\mathbb{P}$ nef
$g = 0$ local $GW$ inv.
local A-model

a family of affine curves
$C_\alpha \subset \mathbb{T}^2$
VMHS on $H^2(\mathbb{T}^2, C_\alpha)$
local B-model
Why $H^2(\mathbb{T}^2, C^\circ_a)$, Not $H^1(C^\circ_a)$?

1. $H^2(\mathbb{T}^2, C^\circ_a)$ has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

- **VHS on $H^3$**
  - Hodge filtration $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$
  - has dim $F^3 = 1$
  - $H^3$ generated by holo. 3-form $\in F^3$
  - and GM connection

- **VMHS on $H^2(\mathbb{T}^2, C^\circ_a)$**
  - Hodge filtration $0 \subset F^2 \subset F^1 \subset F^0 = H^2$
  - has dim $F^2 = 1$
  - $H^2$ generated by
    
    $\left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \in F^2$
    
    and GM connection

2. Period integrals of $\left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$ satisfy the A-hypergeometric system.

3. $H^1(C^\circ_a)$ does not have these properties!
Comparison with Mirror Symmetry

**Mirror Symmetry**
- **A-model**
  - GW inv of $X$
- **B-model**
  - VHS on $H^3(X^\vee)$
  - holo. 3-form $\Omega$
  - Yukawa coupling:
    $$\int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega$$
  - holo. anomaly eq.

**Local Mirror Symmetry**
- local GW inv of $\mathbb{P}$
- VMHS on $H^2(\mathbb{T}^2, C_3)$
  - $\omega := \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$
    - $?$
    - $??$
Our goal

- Want to define analogues of the Yukawa coupling, BCOV’s holomorphic anomaly equation for local B-model.

- $H^2(\mathbb{T}^2, C_a)$ and its variation of mixed Hodge structures were determined by Batyrev(1993) and Stienstra(1997). This enables us to give a definition of Yukawa coupling for local B-model.

- In several examples of local B-model, the Yukawa couplings have been computed [Klemm–Zaslow, Jinzenji–Forbes, Aganagic–Bouchard–Klemm, Haghihat–Klemm–Rauch, Alim–Länge-Mayr, Brini–Tanzini] without a direct definition. Our definition agrees with these results.

- We also proposed how to modify Bershadsky–Cecotti–Ooguri–Vafa’s holomorphic anomaly equation to the setting of local B-model.
In the following, I will explain the case

\[ \Delta = (-1,0), (0,1), (1,0), (-1,-1) \]

- Local A-model: GW invariants of the Hirzebruch surface \( \mathbb{F}_1 \).
- Local B-model:
  - \( \Delta \leadsto \) the Laurent polynomial:
    \[
    F_a(t_1, t_2) := a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1} + \frac{a_4}{t_1 t_2}
    \]
    (Integral points in \( \Delta \) \( \iff \) monomials)
  - \( F_a(t_1, t_2) \leadsto \) the affine curve in \( \mathbb{T}^2 := (\mathbb{C}^*)^2 \):
    \[
    C_a := \{(t_1, t_2) \mid F_a(t_1, t_2) = 0\} = \text{genus one curve} - 4 \text{ points}
    \]
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Decription of $H^2(\mathbb{T}^2, C_{a}^\circ)$ [Batyrev, Stienstra]

$\Delta, F_a \sim R_{F_a} \sim H^2(\mathbb{T}^2, C_{a}^\circ)$

“Jacobian ring”
“Jacobian Ring” \( \mathcal{R}_{F_a} \)

- \( \Delta(k) : \Delta \) enlarged by \( k \)-times
  
  \[ \Delta(0) \quad \Delta(1) = \Delta \quad \Delta(2) \]

- Consider the ring:
  
  \[ S_\Delta := \bigoplus_{k \geq 0} \bigoplus_{(m_1, m_2) \in \Delta(k)} \mathbb{C} t_0^k t_1^{m_1} t_2^{m_2} \subset \mathbb{C}[t_0, t_1^{\pm 1}, t_2^{\pm 1}] \]

- Let \( \mathcal{D}_i \ (i = 0, 1, 2) \) be derivations:
  
  \[ \mathcal{D}_0(t_0^k t_1^{m_1} t_2^{m_2}) = (k + t_0 F_a(t)) t_0^k t_1^{m_1} t_2^{m_2} \]
  
  \[ \mathcal{D}_i(t_0^k t_1^{m_1} t_2^{m_2}) = (m_i + t_0 t_i \partial_i F_a(t)) t_0^k t_1^{m_1} t_2^{m_2} \quad (i = 1, 2) \]

- \( \mathcal{R}_{F_a} \) is the quotient vector space given by:
  
  \[ \mathcal{R}_{F_a} := S_\Delta / \left( \sum_{i=0}^{2} \mathcal{D}_i S_\Delta \right) \]
Let’s compute \( R_{F_a} \)

\[
S_\Delta = \left( C1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0t_1 \oplus \mathbb{C}t_0t_2 \oplus \mathbb{C}\frac{t_0}{t_1} \oplus \mathbb{C}\frac{t_0}{t_1t_2} \oplus \mathbb{C}t_0^2 \oplus \ldots \oplus \ldots \right)_{\Delta(0)} \oplus \left( \mathbb{C}t_0 \oplus \mathbb{C}\frac{t_0}{t_1} \oplus \mathbb{C}\frac{t_0}{t_1t_2} \oplus \mathbb{C}t_0^2 \oplus \ldots \right)_{\Delta(1)} \oplus \left( \mathbb{C}t_0^2 \right)_{\Delta(2)}
\]

- Relations \( D_i 1 = 0 \ (i = 0, 1, 2) \) imply:

\[
t_0t_1 = -2\frac{a_0}{a_1}t_0 - \frac{a_2}{a_0}t_0t_2 \quad , \quad \frac{t_0}{t_1} = -\frac{a_1}{a_3}t_0 - \frac{a_2}{a_3}t_0t_2 \quad , \quad \frac{t_0}{t_1t_2} = \frac{a_2}{a_4}t_0t_2
\]

- Relations \( D_i t_0^k t^m \ (k \geq 1, \ m \in \Delta(k)) \) imply

\[
t_0^{k+1} t^m = \text{const.} t_0^2 + \text{term of } t_0\text{-degree } 1 \quad (k \geq 1, \ m \in \Delta(k + 1))
\]

\[
\therefore \ R_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0t_2 \oplus \mathbb{C}t_0^2
\]

- For a general 2-dim reflexive polyhedron \( \Delta \),

\[
\dim R_{F_a} = \text{volume of } \Delta = \#\text{integral points of } \Delta
\]
Isomorphism $\mathcal{R}_{F_a} \cong H^2(\mathbb{T}^2, C_a^\circ)$

- Note that there is an exact sequence

$$0 \rightarrow PH^1(C_a^\circ) \rightarrow H^2(\mathbb{T}^2, C_a^\circ) \rightarrow H^2(\mathbb{T}^2) \rightarrow 0$$

$(PH^1(C_a^\circ) := H^1(C_a^\circ)/H^1(\mathbb{T}^2))$

- An isomorphism $\mathcal{R}_{F_a} \sim \rightarrow H^2(\mathbb{T}^2, C_a^\circ)$

$$\mathbb{C}1 \leftrightarrow H^2(\mathbb{T}^2) : 1 \leftrightarrow \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$$

$$\mathbb{C}t_0 \oplus \mathbb{C}t_0 \mathbb{t}_2 \oplus \mathbb{C}t_0^2 \leftrightarrow PH^1(C_a^\circ) : t_0^k t^m \leftrightarrow \left( 0, \text{Res}_{F_a=0} \frac{(k-1)! t^m}{(-1)^{k-1} F_a^k} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \right)$$

- More explicitly: $(C_a$ is a compactification of $C_a^\circ)$

$$\mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0 \mathbb{t}_2 \oplus \mathbb{C}t_0^2 \uparrow \updownarrow$$

$$\begin{array}{c}
\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \\
\text{on } \mathbb{T}^2 \\
\hline
\end{array}$$

$$\begin{array}{c}
(1,0)-\text{form} \\
\text{on } C_a \\
\hline
(1,0)-\text{form} \\
\text{on } C_a \text{ with poles} \\
\hline
(0,1)-\text{form} \\
\text{on } C_a
\end{array}$$
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What’s Mixed Hodge Structure?

- $H^k(V)$ of a smooth projective variety $V$ has the canonical Hodge structure of weight $k$:

$$H^k(V) = \bigoplus_{p+q=k} H^{p,q}(V) \quad \text{(Hodge decomposition)}$$

- To generalize this to $H^k(U)$ of an open variety $U$, it is necessary to consider the mixed Hodge structure.

- Roughly speaking, the mixed Hodge structure is the direct sum of Hodge structures of different weights:

$$\bigoplus_{l} \bigoplus_{p+q=l} H^{p,q}$$
Definition of MHS

- Mixed Hodge structure of weight $k$ consists of:
  - free abelian group $H_\mathbb{Z}$,
  - the weight filtration $W_\bullet$ on $H_\mathbb{Z}$ (increasing filtration),
  - the Hodge filtration $F^\bullet$ on $H_\mathbb{C}$ (decreasing filtration),

such that the induced Hodge filtration on $W_l/W_{l-1}$ has a Hodge structure of weight $l + k$.

$$H^{p,k+l-p} := \frac{F^p W_l/W_{l-1}}{F^{p+1} W_l/W_{l-1}}$$

satisfy $H^{p,q} = \overline{H}^{q,p}$. 
MHS for an open variety

- If $U = V - D$ where $V$ is a smooth projective variety and $D$ is a simple normal crossing divisor, then $H^k(U)$ has a canonical mixed Hodge structure.
- Hodge filtration $F^\bullet$ is induced from the filtration on $\Omega^\bullet_V(\log D)$

$$F^p\Omega^\bullet_V(\log D) = \Omega^{\geq p}_V(\log D)$$

- Weight filtration is induced from the filtration

$$W_1\Omega^\bullet_V(\log D) = \bigwedge^l \Omega^1_V(\log D) \wedge \Omega^{-l}_V.$$ 

Roughly speaking, $W_{k+l} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

- For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

$$\ldots \longrightarrow H^k(U_1) \longrightarrow H^{k+1}(U_2, U_1) \longrightarrow H^{k+1}(U_2) \longrightarrow \ldots$$

is a long exact sequence of MHS’s.
MHS on $H^2(\mathbb{T}^2, C^\circ_a) \cong \mathcal{R}_{F_a}$

- Hodge filtration $\iff t_0$-grading
- Weight filtration $\iff$ weight of $t_0^k t^m$ depends on whether $m \in \Delta(k)$ is a vertex or not.

<table>
<thead>
<tr>
<th>$t_0$-deg. Hodge filt.</th>
<th>(0) $F^2$</th>
<th>(1) $F^1/F^2$</th>
<th>(2) $F^0/F^1$</th>
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<tbody>
<tr>
<td>Weight filt.</td>
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<tr>
<td>$W_1$</td>
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<tr>
<td>$W_2/W_1$</td>
<td>$H^1(C_a)$</td>
<td>$\mathbb{C}t_0$</td>
<td>$\mathbb{C}t_0^2$</td>
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<td>$W_3/W_2$</td>
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<tr>
<td>$W_4/W_3$</td>
<td>$\mathbb{C}1$</td>
<td>$dt_1/t_1 \wedge dt_2/t_2$ on $\mathbb{T}^2$</td>
<td>$\mathbb{C}1$</td>
</tr>
</tbody>
</table>

$(1, 0)$-form on $C_a$

$(0, 1)$-form on $C_a$

$(1, 0)$-form on $C_a$ with poles at $C_a - C^\circ_a$
So far, the parameter $a$ of $C_a^\circ$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.

$F_a(t)$ is $\Delta$-regular $\iff$

$$a_1 a_2 a_3 a_4 \neq 0, \quad a_3(a_0^2 - 4a_1 a_3)^2 - a_2 a_4(a_0^3 - 36a_0 a_1 a_3 + 27a_1 a_2 a_4) \neq 0$$

The Gauss–Manin connection $\nabla$ corresponds to the differential operators on $\mathcal{R}_{F_a} \otimes \mathbb{C}(a)$ [Batyrev, Stienstra]:

$$\nabla \partial_{am} =: \nabla a_m \iff D_{am} := \partial_{am} + t_0 t^m \quad (m \in \Delta)$$
**Gauss–Manin connection II**

How $\nabla_{a_m}$ acts on $H^2(\mathbb{T}^2, C^o_a) \cong \mathcal{R}_{F_a}$:

<table>
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<tr>
<th></th>
<th>$F^2$</th>
<th>$F^1/F^2$</th>
<th>$F^0/F^1$</th>
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<tbody>
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<td>$W_1$</td>
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<tr>
<td>$W_4/W_3$</td>
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</table>

By the GM connection $\nabla$,  
- The Weight filtration is preserved: $\nabla_{a_m} W_k \subset W_k$  
- The Hodge filtration is changed by $1$ (Griffiths transversality): $\nabla_{a_m} F^k \subset F^{k+1}$
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Yukawa coupling for B-model

- In the case of $H^3(X^\vee)$ of a Calabi–Yau threefold $X^\vee$, the Yukawa coupling is the $\mathcal{O}_\mathcal{M}$-linear map:

$$\mathcal{T}\mathcal{M}^3 \to \mathcal{O}_\mathcal{M}; \langle \partial_i, \partial_j, \partial_k \rangle = \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega$$

($\mathcal{M}$ is the complex moduli space, $\Omega$ is a relative holomorphic 3-form.)

- We want to devise this definition for $H^2(\mathbb{T}^2, C_{a^\circ})$.

- In this definition, the polarization

$$H^3(X^\vee) \times H^3(X^\vee) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{X^\vee} \alpha \wedge \beta$$

is necessary.

For $H^2(\mathbb{T}^2, C_{a^\circ})$, our idea is to use the polarization of

$$\mathcal{W}_1 H^2(\mathbb{T}^2, C_{a^\circ}) = H^1(C_a)$$

instead.
How can we define the Yukawa coupling?

Recall

<table>
<thead>
<tr>
<th>$H^1(C_a) = W_1$</th>
<th>$F^2$</th>
<th>$F^1/F^2$</th>
<th>$F^0/F^1$</th>
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<td>$W_2/W_1$</td>
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<td>$W_3/W_2$</td>
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<tr>
<td>$W_4/W_3$</td>
<td>$\mathbb{C}^1$</td>
<td>$\nabla a_0$</td>
<td>$\omega := (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in H^2(\mathbb{T}^2, C^\circ_a)$</td>
</tr>
</tbody>
</table>

Therefore $-\sqrt{-1} \int_{C_a} \nabla^2_{a_0} \omega \wedge \nabla a_0 \omega$ is well-defined.

We define this as the Yukawa coupling $\langle \partial a_0, \partial a_0; \partial a_0 \rangle$
Definition of Yukawa coupling for $H^2(\mathbb{T}^2, C_a^\circ)$

1. $\mathbb{L}$: the base space of the family:
   
   $$\mathbb{L} := \{a \in \mathbb{C}^5 \mid a_1 a_2 a_3 a_4 \neq 0, a_3 (a_0^2 - 4a_1 a_3)^2 - a_2 a_4 (a_0^3 - 36a_0 a_1 a_3 + 27a_1 a_2 a_4) \neq 0\}$$

2. $T^0\mathbb{L}$: the rank 1 subbundle of $T\mathbb{L}$ spanned by $\partial_{a_0}$

3. The Yukawa coupling is a multilinear map:

   $$T\mathbb{L} \times T\mathbb{L} \times T^0\mathbb{L} \to \mathcal{O}_{\mathbb{L}}; \langle A, B; C \rangle := -\sqrt{-1} \int_{C_a} (\nabla_A \nabla_B \omega)' \wedge \nabla_C \omega$$

   - $\nabla_C \omega \in F^1 \cap \mathcal{W}_1$ is a $(1, 0)$-form on $C_a$
   - $\nabla_A \nabla_B \omega$ may be outside of $\mathcal{W}_1$. But such a class can be written as
     
     $$\nabla_A \nabla_B \omega = \alpha_1 \quad (1, 0)\text{-form on } C_a \text{ with poles}$$
     
     $$+ \alpha_2 \quad (0, 1)\text{-form on } C_a \text{ (without poles)}$$

   So set $(\nabla_A \nabla_B \omega)' := \alpha_2$.  

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Computation of Yukawa coupling

- We can compute the Yukawa coupling by solving differential equations coming from A-hypergeometric system.

\[
\langle \partial a_0, \partial a_0; \partial a_0 \rangle = \frac{1}{a_0^3} \frac{(8 - 9z_2)}{d(z_1, z_2)},
\]

\[
\langle \partial a_1, \partial a_1; \partial a_0 \rangle = \frac{1}{a_0 a_1^2} \frac{1 + 4z_1 - z_2 - 3z_1 z_2}{d(z_1, z_2)},
\]

\[
\langle \partial a_1, \partial a_2; \partial a_0 \rangle = \frac{1}{a_0 a_1 a_2} \frac{1 - 4z_1 - z_2 + 6z_1 z_2}{d(z_1, z_2)},
\]

\[
\langle \partial a_2, \partial a_2; \partial a_0 \rangle = \frac{1}{a_0 a_2^2} \frac{z_2 (1 + 12z_1)}{d(z_1, z_2)}, \quad \ldots
\]

\[
z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_0 a_3},
\]

\[
d(z_1, z_2) = (1 - 4z_1)^2 - z_2 (1 - 36z_1 + 27z_1 z_2).
\]
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Holomorphic Anomaly Eq. (for B-model)

- In the B-model of mirror symmetry, there is Bershadsky–Cecotti–Ooguri–Vafa’s (BCOV’s) holomorphic anomaly equation. It is a system of differential equations for the B-model topological string amplitude $F_g$ ($g \geq 1$).
- Let $\mathcal{M}$ be the complex moduli space of a Calabi–Yau 3-fold $X^\vee$, and $z_1, \ldots, z_n$ be local coordinates of $\mathcal{M}$
- BCOV’s holomorphic anomaly eq. involves:
  - Kähler potential of $\mathcal{M}$: $K = -\log \sqrt{-1} \int_{X^\vee} \Omega \wedge \bar{\Omega}$
  - Kähler metric on $\mathcal{M}$: $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$
  - Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes 3)$
- BCOV’s holomorphic anomaly equation:

$$\bar{\partial}_i F_g = \frac{1}{2} \sum_{j, k, \bar{j}, \bar{k}} \overline{C}_{i\bar{j}k} e^{2K} G^{i\bar{j}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})$$

- We want to formulate BCOV’s holomorphic anomaly equation for local B-model.
Quotient family

- To formulate the holomorphic anomaly equation for local B-model, it is convenient to introduce the quotient family of our family of affine curves by the $\mathbb{T}^3$-action:

\[
\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2).
\]

\[
C := \left\{ (t, z) \in \mathbb{T}^2 \times \mathcal{M} \mid 1 + z_1 t_1 + t_2 + \frac{1}{t_1} + \frac{z_2}{t_1 t_2} = 0 \right\}
\]

\[
\downarrow \quad \left( z_1 := \frac{a_1 a_3}{a_0^2}, \; z_2 = \frac{a_2 a_4}{a_0 a_3} \right)
\]

\[
\mathcal{M} := \mathbb{L}' / \mathbb{T}^3
\]

\[
= \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 \neq 0, (1 - 4z_1)^2 - z_2(1 - 36z_1 + 27z_1 z_2) \neq 0 \right\}
\]

\[
(\mathbb{L}' := \left\{ a \in \mathbb{C}^5 \mid a_0 a_1 a_2 a_3 a_4 \neq 0,
\quad a_3(a_0^2 - 4a_1 a_3)^2 - a_2 a_4(a_0^3 - 36a_0 a_1 a_3 + 27a_1 a_2 a_4) \neq 0 \right\})
\]
An Analogue of Special Kähler geometry

- $z_1, z_2$: the coordinates of $\mathcal{M}$, $\partial_i := \partial_{z_i}$ ($i = 1, 2$)
- $T^0\mathcal{M} \subset T\mathcal{M}$ = the rank 1 subbundle spanned $\theta_0$:
  \[
  \theta_0 = \text{the pushforward of the vector field } a_0 \partial_{a_0} \text{ on } \mathbb{L}'
  \]
- The Yukawa couplings: (The above definition is also valid for the quotient family.)
  \[
  Y_{i,j;0} := \langle \theta_i, \theta_j; \theta_0 \rangle \quad (\theta_i := z_i \partial_i)
  \]
- Hermitian metric on $T^0\mathcal{M}$:
  \[
  (\theta_0, \theta_0) = \sqrt{-1} \int_{C_{\theta_0}} \nabla_{\theta_0} \omega \wedge \overline{\nabla_{\theta_0} \omega} =: G_{0\bar{0}}
  \]
- $Y_{i,j;0}$ and $G_{0\bar{0}}$ satisfy the relation:
  \[
  \overline{\partial_j \theta_i} \frac{G_{0\bar{0}}}{G_{0\bar{0}}} = - \frac{Y_{i0;0}}{G_{0\bar{0}}} \overline{Y_{j0;0}} \quad (\overline{\partial_j} = \overline{z_j} \partial_{\overline{z}_j})
  \]
Holomorphic Anomaly Eq.

- Let $\tilde{C}^{(g)}_n$ be the “B-model topological string amplitude”
- Set
  $$\tilde{C}^{(0)}_0 = \tilde{C}^{(0)}_1 = \tilde{C}^{(0)}_2 = 0, \quad \tilde{C}^{(0)}_3 = Y_{00;0}$$
  $$\tilde{C}^{(1)}_0 = 0$$
  $$\tilde{C}^{(g)}_{n+1} = \left( \theta_0 - n \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}} \right) \tilde{C}^{(g)}_n$$

- Holomorphic anomaly equation for local B-model we propose is

  $$\bar{\theta}_j \tilde{C}^{(1)}_1 = -\frac{1}{2} \bar{\theta}_j \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}}$$
  $$\bar{\theta}_j \tilde{C}^{(g)}_0 = \frac{Y_{j,0;0}}{2 G_{0\bar{0}}^2} (\tilde{C}^{(g-1)}_2 + \sum_{r=1}^{g-1} \tilde{C}^{(r)}_1 \tilde{C}^{(g-r)}_1) \quad (g \geq 2)$$
Remarks

- This holo. anomaly eq. is consistent with the following observations made previously by several authors [Klemm–Zaslow, Hosono, Haghhiat–Klemm–Rauch, Aganagic–Bouchard–Klemm, Alim–Länge–Mayr].
  - Contains no Kähler potential;
  - Only the rank 1 subbundle $T^0 \mathcal{M}$ of $T \mathcal{M}$ matters. Similar to the one-parameter model.

- Holomorphic anomaly equation can be solved by using the Feynman diagrams (with only one propagator) and also by Yamaguchi–Yau’s method.
Solutions

- $\tilde{C}_n^{(g)}$ is a polynomial of degree $3g - 3 + n$ in $\mathbb{C}(z)[A]$, with
  \[ A = \frac{\theta_0 G_{0\bar{0}}}{G_{0\bar{0}}}. \]

- For $(g, n) = (1, 1), (2, 0)$,
  \[ \tilde{C}_1^{(1)} = \frac{1}{2} A + f_1^1(z) \]
  \[ \tilde{C}_0^{(2)} = \frac{-1}{2Y_{00;0}} \left[ \frac{5}{12} A^3 - \frac{\theta_0 Y_{00;0}}{4Y_{00;0}} A^2 + \left( \theta_0 f_1^1(z) + (f_1^1(z))^2 - \frac{\kappa(z)}{2} \right) A \right] + f_0^2(z) \]

- where
  - $\kappa(z)$ is determined from special geometry relation:
    \[ \kappa(z) = \frac{2z_1(-32 + 192z_1 + 282z_2 - 144z_1 z_2 - 486z_2^2 + 243z_2^3)}{(1 - 4z_1)^2 - z_2(1 - 36z_1 + 27z_1 z_2)} \]
  - $f_1^1(z), f_0^2(z)$ are holomorphic functions on $\mathcal{M}$ which cannot be determined from the holomorphic anomaly eq. (holomorphic ambiguities)
Outline

1. Motivation
2. Jacobian ring description of $H^2(\mathbb{T}^2, C^\circ_\alpha)$
3. Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^\circ_\alpha)$
4. Yukawa coupling
5. Holomorphic anomaly equation
6. Witten's Geometric Quantization Approach
7. Appendix (Local GW invariant)
Witten’s Approach

- Witten gave a heuristic interpretation of BCOV’s holomorphic anomaly equation in the framework of geometric quantization ('93):
  
  | Holomorphic anomaly eq. | A proj. flat connection on infinite dim. vector bundle $\mathcal{H} \to \mathcal{M}$ |

- His approach can be fit into the setting of local B-model and our holomorphic anomaly equation.
Applying Geometric Quantization

- Consider the symplectic vector space $W = H^1(C_z, \mathbb{R})$ with the symplectic form given by the cup product.
- For each parameter $z \in \mathcal{M}$, let $x, \bar{x}$ be the complex coordinates of $W_\mathbb{C}$ associated to the basis
  \[
  \phi = \nabla_{\theta_0} \omega \quad \text{(holo. 1-form),} \quad \bar{\phi}
  \]
- Let $L = W \times \mathbb{C}$ be the complex line bundle with the connection
  \[
  D = \delta + \frac{1}{2} G_{0\bar{0}}(xd\bar{x} - \bar{x}dx).
  \]
  (Remark: $x, \bar{x}$ and $D$ depend on $z \in \mathcal{M}$ !)
- Consider the space of (square integrable) polarized sections
  \[
  \mathcal{H}_z = \{ \Phi \in \Gamma(W, L) \mid D_{\partial_x} \Phi = 0 \} \]
HAE as flat connection

- Witten’s idea is to interpret holomorphic anomaly eq. as a connection on the infinite dimensional vector bundle $\mathcal{H} \rightarrow \mathcal{M}$.
- In the case of local B-model, consider the connection $d$

$$d(\theta_j) = \theta_j$$

$$d(\bar{\theta}_j) = \bar{\theta}_j - \frac{Y_{j0;0}}{2G_{0,\bar{0}}^2} \left( \delta_x - \frac{G_{0\bar{0}}}{2} \bar{x} \right)^2$$

Then $d$ is a projectively flat connection on $\mathcal{H}$!

AND

$$\left\{ \bar{C}_n^{(g)} \right\} \text{ satisfies hol. anomaly eq.} \iff \text{ exp } \left[ \sum_{n,g} \frac{\lambda^{2g-2+n}}{n!} \bar{C}_n^{(g)} x^n \right] \times e^{-\frac{G_{0\bar{0}}}{2} x \bar{x}} \text{ is a flat section}$$
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Local Gromov–Witten invariants

- Genus $g$ local Gromov–Witten invariant $N_{g,\beta}(\mathbb{P})$ of degree $\beta \in H_2(\mathbb{P}, \mathbb{Z})$ is defined by

\[
N_{g,\beta}(\mathbb{P}) = \int_{[\overline{M}_{g,0}(\mathbb{P}, \beta)]^{\text{vir}}} e(\pi_\ast ev_\ast K_\mathbb{P}).
\]

- Here
  - $\overline{M}_{g,n}(\mathbb{P}, \beta)$ is the moduli stack of stable maps to $\mathbb{P}$ of genus $g$ and degree $\beta$,
  - $ev: \overline{M}_{g,1}(\mathbb{P}, \beta) \to \mathbb{P}$ is the evaluation map,
  - $\pi: \overline{M}_{g,1}(\mathbb{P}, \beta) \to \overline{M}_{g,0}(\mathbb{P}, \beta)$ is the map forgetting the marked point,
  - $e$ denotes the Euler class.
- Remark. This is defined for $\beta$ such that $\beta \cdot K_\mathbb{P} \neq 0$. 