Local B-model and Mixed Hodge Structure

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Mirror Symmetry and Gromov–Witten theory
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Local Mirror Symmetry

- Local mirror symmetry was derived by Chiang–Klemm–Yau–Zaslow in 1999.
- It is a statement about local GW invariants of smooth complete weak Fano toric surfaces.
- It is derived from mirror symmetry of toric Calabi–Yau hypersurfaces by considering a certain limits in moduli spaces.
  (e.g. CY hypersurface $\subset \hat{\mathbb{P}}(1, 1, 1, 6, 9) \rightsquigarrow \mathbb{P}^2$)

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Start from:

\[ \Delta \]

2 dim reflexive polyhedron
Reflexive Polyhedra

A reflexive polyhedron $\Delta$ is a polyhedron satisfying:

- it is a convex hull of integral points;
- $0 \in \Delta$;
- Distance between 0 and each codimension 1 face is 1.

There are 16 2-dimensional reflexive polyhedra.
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Local Mirror Symmetry

\[ \Delta \]

2 dim reflexive polyhedron

compact toric surface

\[ \mathbb{P} \quad \text{s.t. } -K_{\mathbb{P}} \text{ nef} \]
Local Mirror Symmetry

\[ \Delta \]

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\[ \mathbb{P} \quad \text{s.t. } -K_{\mathbb{P}} \text{ nef} \]
• Regard integral points of $\Delta$ other than the origin as one dimensional cones in $\mathbb{R}^2$.

• Then they define a complete smooth 2-dimensional fan.

• This fan defines a complete smooth toric surface $\mathbb{P}$ whose anti-canonical divisor is nef.

Example

$\Delta = \triangle v_1 v_2 v_3 \quad \rightarrow \quad \mathbb{P} = \mathbb{P}^2$
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Local Gromov–Witten invariants

Genus $g$ local Gromov–Witten invariant $N_{g,\beta}(\mathbb{P})$ of degree $\beta \in H_2(\mathbb{P}, \mathbb{Z})$ is defined by

$$N_{g,\beta}(\mathbb{P}) = \int_{[\overline{M}_{g,0}(\mathbb{P}, \beta)]^{\text{vir}}} e(\pi_* ev^* K_\mathbb{P}) .$$

- $\overline{M}_{g,n}(\mathbb{P}, \beta)$ is the moduli stack of stable maps to $\mathbb{P}$ of genus $g$ and degree $\beta$,
- $ev : \overline{M}_{g,1}(\mathbb{P}, \beta) \to \mathbb{P}$ is the evaluation map,
- $\pi : \overline{M}_{g,1}(\mathbb{P}, \beta) \to \overline{M}_{g,0}(\mathbb{P}, \beta)$ is the map forgetting the marked point,
- $e$ denotes the Euler class.
- **Remark.** This is defined for $\beta$ such that $\beta \cdot K_\mathbb{P} \neq 0$. 
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compact toric surface

$\mathbb{P}$ s.t. $-K_{\mathbb{P}}$ nef

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local A-model

a family of affine curves

$C^0_a \subset \mathbb{T}^2$

Local B-model and MHS
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Motivation

Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^0_a)$

Yukawa coupling

Holomorphic anomaly equation
Local Mirror Symmetry

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  \[ \text{local A-model} \]
- a family of affine curves
  \[ C_a \subset T^2 \]
\[ \Delta \sim C_a \]

- Associate the Laurent monomial to an integral point in \( \Delta \):
  \[
  (m_1, m_2) \leftrightarrow t^m := \frac{m_1}{t_1} \frac{m_2}{t_2}
  \]

- Take the sum of these monomials with parameters \( a_m \):
  \[
  F_a(t) := \sum_{m \in \Delta} a_m t^m
  \]

Ex. \( \Delta = \bigtriangleup \)

\[
F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.
\]

- The zero set of \( F_a(t) \) is an affine curve in \( \mathbb{T}^2 \):
  \[
  C_a^0 = \{(t_1, t_2) \in \mathbb{T}^2 \mid F_a(t) = 0\}.
  \]

- \( C_a^0 \) = genus 1 complete curve – points

Remark: we must put the \( \Delta \)-regularity condition on \( a_m \) so that \( C_a^0 \) and its compactification are smooth.
\[ \Delta \xrightarrow{\sim} C_a^o \]

- Associate the Laurent monomial to an integral point in \( \Delta \):
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Local Mirror Symmetry

Local A,B-models are related to each other via

\[ A \text{-hypergeometric system with } \beta = \bar{0} \]

\[ \Delta \]

2 dim reflexive polyhedron

Compact toric surface

\[ \mathbb{P} \text{ s.t. } -K_\mathbb{P} \text{ nef} \]

\[ g = 0 \text{ local GW inv.} \]

Local A-model

A family of affine curves

\[ C_\alpha \subset \mathbb{T}^2 \]

VMHS on \( H^2(\mathbb{T}^2, C_\alpha) \)

Local B-model

Studied by Stienstra, Batyrev

Motivation

Mixed Hodge Structure of \( H^2(\mathbb{T}^2, C_\alpha) \)

Yukawa coupling

Holomorphic anomaly equation
Local Mirror Symmetry

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(a system of diff. eqs introduced by Gel’fand–Kapranov–Zelevinsky)

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Local Mirror Symmetry

How are they related?

A-hypergeometric system with \( \beta = \vec{0} \)

Solutions give:
- Mirror map
- A derivative of prepotential

\( \Delta \)
2 dim reflexive polyhedron

compact toric surface
\( \mathbb{P} \) s.t. \(-K_\mathbb{P}\) nef
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\( C_\alpha \subset \mathbb{T}^2 \)
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PF equation for period integrals of "top element"
\( \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \)
Local Mirror Symmetry

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A-hypergeometric system with $\beta = \bar{0}$

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$\triangle$
2 dim reflexive polyhedron

Solutions give:

- Compact toric surface $\mathbb{P}$ s.t. $-K_\mathbb{P}$ nef
  - $g = 0$ local GW inv.
  - local A-model

- A family of affine curves $C_\alpha \subset \mathbb{T}^2$
  - VMHS on $H^2(\mathbb{T}^2, C_\alpha)$
  - local B-model

PF equation for period integrals of "top element" $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$
How are they related?

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\[
\Delta \\
2 \text{ dim reflexive polyhedron}
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A-hypergeometric system with \( \beta = 0 \)

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Local Mirror Symmetry

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VMHS on $H^2(\mathbb{T}^2, C_a)$

local B-model
Why $H^2(\mathbb{T}^2, C_a^\circ)$, Not $H^1(C_a^\circ)$?

(1) It has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

VHS on $H^3$

- Hodge filtration
  $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$
  has dim $F^3 = 1$

- $H^3$ generated by holo. 3-form $\in F^3$
  and GM connection

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Why $H^2(\mathbb{T}^2, C^*_a)$, Not $H^1(C^*_a)$?

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- Hodge filtration
  \[ 0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3 \]
  has dim $F^3 = 1$

- $H^3$ generated by
  holo. 3-form $\in F^3$
  and GM connection

- VMHS on $H^2(\mathbb{T}^2, C^*_a)$
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    \[ 0 \subset F^2 \subset F^1 \subset F^0 = H^2 \]
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    \[ (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in F^2 \]
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(2) Period integrals of $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$ satisfy the A-hypergeometric system.

- $H^1(C^*_a)$ does not have these properties!
Why $H^2(\mathbb{T}^2, C_a^\circ)$, Not $H^1(C_a^\circ)$?

(1) It has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

**VHS on $H^3$**

- Hodge filtration
  
  $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$

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- $H^3$ generated by holo. 3-form $\in F^3$

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**VMHS on $H^2(\mathbb{T}^2, C_a^\circ)$**

- Hodge filtration
  
  $0 \subset F^2 \subset F^1 \subset F^0 = H^2$

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Comparison with Mirror Symmetry

**A**

GW inv of $X$

VHS on $H^3(X^\vee)$

- holo. 3-form $\Omega$
- Yukawa coupling

\[ \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega \]

Important because it is:

- a third derivative of prepotential;
- necessary for BCOV’s holomorphic anomaly eq.

**B**

local GW inv of $\mathbb{P}$

VMHS on $H^2(\mathbb{T}^2, C_3^a)$

- $\omega := \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right)$
- ??
Comparison with Mirror Symmetry

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MS

LMS

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Comparison with Mirror Symmetry

\[ \text{MS} \]

GW inv of \( X \)

VHS on \( H^3(X^\vee) \)

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Our aim

- In several examples of local B-model, the Yukawa couplings have been computed [Klemm–Zaslow, Jinzenji–Forbes, Aganagic–Bouchard–Klemm, Haghhiat–Klemm–Rauch, Alim–Länge-Mayr, Brini–Tanzini]. However, there has been no direct definition.

- We gave a definition of local B-model Yukawa coupling using the results of Batyrev, Stienstra on the VMHS on $H^2(T^2, C_a^\circ)$.
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Plan

- Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^\circ_a)$ [Batyrev, Stienstra]
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Mixed Hodge Structure of $H^2(\mathbb{T}^2, C_a^o)$

- The mixed Hodge structure on $H^2(\mathbb{T}^2, C_a^o)$ was studied by Batyrev (’93) and Stienstra (’97).
- $H^2(\mathbb{T}^2, C_a^o)$ has a Jacobian-ring like description. It is isomorphic to a (quotient) vector space $\mathcal{R}_F$, which is determined by the data of $\Delta$ and $F_a(t)$.
- The variation of Mixed Hodge structures on $H^2(\mathbb{T}^2, C_a^o)$ is also described in terms of $\mathcal{R}_F$. 
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\( \Delta(k) \): the polyhedron obtained by enlarging \( \Delta \) by \( k \)-times.

\[
\begin{align*}
\Delta(0) \quad & \\
\Delta(1) = \Delta \quad & \\
\Delta(2) \quad & \\
\end{align*}
\]
\[ R_F \]

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S_{\Delta}^k := \bigoplus_{m \in \Delta(k)} C t_0^k t^m \quad (t^m := t_1^{m_1} t_2^{m_2})
\]

\[
S_{\Delta} := \bigoplus_{k \geq 0} S_{\Delta}^k, \quad \deg t_0^k t^m := k \quad (a \text{ graded ring})
\]

• Recall the defining equation \( F_a(t) \) of \( C_a^\circ \):

\[
F_a(t) = \sum_{m \in \Delta} a_m t^m.
\]

• Define the differential operators on \( S_{\Delta} \): \( (\theta_x := x \partial_x) \)

\[
\mathcal{D}_0(t_0^k t^m) = (k + t_0 F_a(t)) t_0^k t^m
\]

\[
\mathcal{D}_i(t_0^k t^m) = (m_i + t_0 \theta_{t_i} F_a(t)) t_0^k t^m \quad (i = 1, 2).
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\[
\mathcal{R}_F := S_{\Delta}/(\sum_{i=0}^2 \mathcal{D}_i S_{\Delta})
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\]
Example:

\[ \Delta = \begin{array}{c} \bigtriangleup \end{array} \]

- \[ F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} \]
- Relations \( D_i 1 = 0 \) \((i = 0, 1, 2)\) imply:
  \[
  t_0 t_1 = -\frac{a_0}{3 a_1} t_0, \quad t_0 t_2 = -\frac{a_0}{3 a_2} t_0, \quad \frac{t_0}{t_1 t_2} = -\frac{a_0}{3 a_3} t_0.
  \]

- By similar calculation, an element in \( S^k_\Delta \) \((k \geq 2)\) is equal to
  \[
  \text{const.} t_0^2 + \text{an element in } S^1_\Delta.
  \]

\[ \therefore \mathcal{R}_F \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2. \]

For a reflexive polyhedron \( \Delta \),

\[ \mathcal{R}_F \cong \mathbb{C}1 \oplus R^1_F \oplus \mathbb{C}t_0^2 \]

\[ R^1_F := S^1_\Delta / \mathbb{C}t_0 F_a \oplus \mathbb{C}t_1 \theta t_1 F_a \oplus \mathbb{C} \theta t_2 F_a \]
Example: \[ \Delta = \begin{array}{c} \text{Diagram} \end{array} \]

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Example: $\Delta = \begin{array}{c}
\end{array}$

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\[
\begin{align*}
    t_0 t_1 &= - \frac{a_0}{3a_1} t_0, \\
    t_0 t_2 &= - \frac{a_0}{3a_2} t_0, \\
    \frac{t_0}{t_1 t_2} &= - \frac{a_0}{3a_3} t_0.
\end{align*}
\]

- By similar calculation, an element in $S^k_\Delta$ ($k \geq 2$) is equal to

\[
\text{const. } t^2 + \text{an element in } S^1_\Delta.
\]

\[\therefore R_F \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t^2_0.\]

For a reflexive polyhedron $\Delta$,

\[R_F \cong \mathbb{C}1 \oplus R^1_F \oplus \mathbb{C}t^2_0\]

\[R^1_F := S^1_\Delta / \mathbb{C}t_0 F_a \oplus \mathbb{C}t_1 \theta t_1 F_a \oplus \mathbb{C}\theta t_2 F_a\]
Example: \( \Delta = \text{\image} \)

- \( F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} \)

- Relations \( D_i 1 = 0 \) (\( i = 0, 1, 2 \)) imply:
  
  \[
  t_0 t_1 = -\frac{a_0}{3 a_1} t_0, \quad t_0 t_2 = -\frac{a_0}{3 a_2} t_0, \quad \frac{t_0}{t_1 t_2} = -\frac{a_0}{3 a_3} t_0.
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\]

- By similar calculation, an element in \( S_{\Delta}^k \) \((k \geq 2)\) is equal to

\[
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\]

\[
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\]

For a reflexive polyhedron \( \Delta \),

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R_F \cong \mathbb{C} t_0 \oplus R_F^1 \oplus \mathbb{C} t_0^2
\]

\[
R_F^1 := S_{\Delta}^1 / \mathbb{C} t_0 F_a \oplus \mathbb{C} t_1 \theta_{t_1} F_a \oplus \mathbb{C} \theta_{t_2} F_a
\]
Example:

- $F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}$
- Relations $\mathcal{D}_i 1 = 0$ $(i = 0, 1, 2)$ imply:

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t_0 t_1 = -\frac{a_0}{3a_1} t_0, \quad t_0 t_2 = -\frac{a_0}{3a_2} t_0, \quad \frac{t_0}{t_1 t_2} = -\frac{a_0}{3a_3} t_0.
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For a reflexive polyhedron $\Delta$,

\[ \mathcal{R}_F \cong \mathbb{C} 1 \oplus R^1_F \oplus \mathbb{C} t_0^2 \]

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\[ \mathcal{R}_F \cong H^2(\mathbb{T}^2, C^\circ_a) \]

Stienstra, Batyrev showed that \( \mathcal{R}_F \cong H^2(\mathbb{T}^2, C^\circ_a) \). This isomorphism is as follows.

- Note that
  \[ \mathcal{R}_F \cong \mathcal{R}_F^+ \oplus \mathbb{C}1 \]
  (\( \mathcal{R}_F^+ \subset \mathcal{R}_F \): spanned by monomials \( t_0^k t^m \) with \( k \geq 1 \))
- There is an exact sequence
  \[
  0 \to PH^1(C^\circ_a) \to H^2(\mathbb{T}^2, C^\circ_a) \to H^2(\mathbb{T}^2) \to 0
  \]
  \( (PH^1(C^\circ_a) \coloneqq H^1(C^\circ_a)/H^1(\mathbb{T}^2)) \)

\[ \mathbb{C}1 \iff H^2(\mathbb{T}^2) \text{-part} : \quad 1 \iff (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \]

\[ \mathcal{R}_F^+ \iff PH^1(C^\circ_a) \text{-part} : \quad t_0^k t^m \iff (0, \text{Res}_{F_a=0} \frac{(-1)^{k-1}(k-1)!}{F_a^k} t^m \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}) \]
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- There is an exact sequence
  \[ 0 \longrightarrow PH^1(C_a^\circ) \longrightarrow H^2(\mathbb{T}^2, C_a^\circ) \longrightarrow H^2(\mathbb{T}^2) \longrightarrow 0 \]
  (\( PH^1(C_a^\circ) := H^1(C_a^\circ)/H^1(\mathbb{T}^2) \))

\[ \mathbb{C}1 \leftrightarrow H^2(\mathbb{T}^2) \text{-part} : \quad 1 \leftrightarrow \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \]
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\]
Example

\[ \Delta = \ \ \ \ \ \ \]  

- \[ \mathcal{R}_F \cong \mathbb{C}^1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \]

- \[ \text{PH}^1(C_\alpha) = H^1(C_a) \text{ (} C_a \text{ is a compactification of } C_\alpha \text{)} \]

\[ 1 \leftrightarrow \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \text{ on } \mathbb{T}^2 \]

\[ t_0 \leftrightarrow (1, 0)-\text{form on } C_a \]

\[ t_0^2 \leftrightarrow (0, 1)-\text{form on } C_a \]
Example

\[ \Delta = \begin{array}{c}
\end{array} \]

\[ \mathcal{R}_F \cong \bigoplus_{H^2(T^2)} \mathbb{C} \bigoplus \mathbb{C}t_0 \bigoplus \mathbb{C}t_0^2 \bigoplus_{PH^1(C_a^0)} \]

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Example

\[ \Delta = \bigtriangleup \]

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\[
\begin{align*}
1 & \iff \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \text{ on } \mathbb{T}^2 \\
t_0 & \iff (1, 0)-\text{form on } C_a \\
t_0^2 & \iff (0, 1)-\text{form on } C_a
\end{align*}
\]
Example

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What’s Mixed Hodge Structure?

- $H^k(V)$ of a smooth projective variety $V$ has the canonical Hodge structure of weight $k$:

$$H^{p,q} = H^{p,q}(V) \quad \text{(Hodge decomposition)}$$

- Mixed Hodge structure is, in a sense, a generalization of Hodge structure to $H^k(U)$ of an open variety $U$.
- Mixed Hodge structure of weight $k$ consists of
  - free abelian group $H_\mathbb{Z}$,
  - the weight filtration $W_\bullet$ on $H_\mathbb{Z}$ (increasing filtration),
  - the Hodge filtration $F^\bullet$ on $H_\mathbb{C}$ (decreasing filtration),

such that the induced Hodge filtration on $W_i/W_{i-1}$ has a Hodge structure of weight $l + k$.

$$H^{p,k+l-p} := \frac{F^p W_i/W_{i-1}}{F^{p+1} W_i/W_{i-1}} \quad \text{satisfy} \quad H^{p,q} = H^{q,p}_{\overline{}}.$$
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• If $U = V - D$ where $V$ is a smooth projective variety and $D$ is a simple normal crossing divisor, then $H^k(U)$ has a canonical mixed Hodge structure.

• Hodge filtration $F^\bullet$ is induced from the filtration on $\Omega^\bullet_V(\log D)$

\[ F^p \Omega^\bullet_V(\log D) = \Omega^{\geq p}_V(\log D) \]

• Weight filtration is induced from the filtration

\[ W_l \Omega^\bullet_V(\log D) = \wedge^l \Omega^1_V(\log D) \wedge \Omega^{*-l}_V. \]

Roughly speaking, $W_{k+l} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

• For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

\[ \ldots \rightarrow H^k(U_1) \rightarrow H^{k+1}(U_2, U_1) \rightarrow H^{k+1}(U_2) \rightarrow \ldots \]

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$$F^p \Omega^\bullet_V(\log D) = \Omega^\succ^p_V(\log D)$$

• Weight filtration is induced from the filtration

$$W_i \Omega^\bullet_V(\log D) = \wedge^i \Omega^1_V(\log D) \wedge \Omega^{-i}.$$  

Roughly speaking, $W_{k+1} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

• For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

$$\ldots \longrightarrow H^k(U_1) \longrightarrow H^{k+1}(U_2, U_1) \longrightarrow H^{k+1}(U_2) \longrightarrow \ldots$$

is a long exact sequence of MHS’s.
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Hodge filtration

Hodge filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ is given by the filtration on $\mathcal{R}_F$:

- Let $\mathcal{E}^{-i}$ ($i = 0, 1, 2, \ldots$) be the subspace of $\mathcal{R}_F$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.

![Diagram](image)

- $\mathcal{E}^0 = \mathbb{C}1 \Leftrightarrow H^2(\mathbb{T}^2)$, $\mathcal{R}_F = \mathcal{E}^{-2} = \mathcal{E}^{-3} = \mathcal{E}^{-4} = \ldots$

<table>
<thead>
<tr>
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<table>
<thead>
<tr>
<th>0 $\subset$ $\mathcal{E}^0$ $\subset$ $\mathcal{E}^{-1}$ $\subset$ $\mathcal{E}^{-2}$ $=\mathcal{R}_F$</th>
</tr>
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<tr>
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</table>

$H^2(\mathbb{T}^2, C_a^\circ)$
Hodge filtration

Hodge filtration on $H^2(\mathbb{T}^2, C_\alpha^\circ)$ is given by the filtration on $\mathcal{R}_F$:

- Let $\mathcal{E}^{-i} (i = 0, 1, 2, \ldots)$ be the subspace of $\mathcal{R}_F$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.

Ex.

$\mathcal{E}^0$  $\mathcal{E}^{-1}$  $\mathcal{E}^{-2}$

- $\mathcal{E}^0 = \mathbb{C}1 \Leftrightarrow H^2(\mathbb{T}^2), \mathcal{R}_F = \mathcal{E}^{-2} = \mathcal{E}^{-3} = \mathcal{E}^{-4} = \ldots$

\[
\begin{array}{cccc}
0 & \subset & \mathcal{E}^0 & \subset \mathcal{E}^{-1} & \subset \mathcal{E}^{-2} & = & \mathcal{R}_F \\
& | & | & | & | & | & \\
0 & \subset & F^2 & \subset F^1 & \subset F^0 & = & H^2(\mathbb{T}^2, C_\alpha^\circ)
\end{array}
\]
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Ex.

$\mathcal{E}^0$  \hspace{1cm} $\mathcal{E}^{-1}$  \hspace{1cm} $\mathcal{E}^{-2}$

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<td>$F^1$</td>
<td>$F^0$</td>
<td>$H^2(\mathbb{T}^2, C_a^\circ)$</td>
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</tbody>
</table>
Weight filtration

Weight filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ is given by the following.

- Let $l_j$ (1 ≤ j ≤ 3) be the subspace of $\mathcal{R}_F$ spanned by the images of monomials $t_0^k t^m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $l_4 := \mathcal{R}_F$.

Ex.

- $l_3 = \mathcal{R}_F^\perp \Leftrightarrow PH^1(C_a^\circ)$

<table>
<thead>
<tr>
<th>0 ⊂ l_1 ⊂ l_2 ⊂ l_3 ⊂ l_4 = R_F</th>
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<tr>
<td>0 ⊂ W_1 ⊂ W_2 = W_3 ⊂ W_4 = H^2(\mathbb{T}^2, C_a^\circ)</td>
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</tbody>
</table>
Weight filtration

Weight filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ is given by the following.

- Let $I_j (1 \leq j \leq 3)$ be the subspace of $R_F$ spanned by the images of monomials $t_0^k t_m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $I_4 := R_F$.

\[ 0 \subset I_1 \subset I_2 \subset I_3 \subset I_4 = R_F \]

\[ 0 \subset W_1 \subset W_2 = W_3 \subset W_4 = H^2(\mathbb{T}^2, C_a^\circ) \]
Weight filtration

Weight filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ is given by the following.

- Let $l_j (1 \leq j \leq 3)$ be the subspace of $\mathcal{R}_F$ spanned by the images of monomials $t_0^k t^m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $l_4 := \mathcal{R}_F$.

Ex.

- $l_3 = \mathcal{R}_F^+ \iff \text{PH}^1(C_a^\circ)$

<table>
<thead>
<tr>
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<td>$0 \subset W_1 \subset W_2 = W_3 \subset W_4 = H^2(\mathbb{T}^2, C_a^\circ)$</td>
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Weight filtration

Weight filtration on $H^2(\mathbb{T}^2, C^\circ_a)$ is given by the following.

- Let $l_j \ (1 \leq j \leq 3)$ be the subspace of $\mathcal{R}_F$ spanned by the images of monomials $t_0^k t_m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $l_4 := \mathcal{R}_F$.

![Diagram](image)

Ex.

- $l_3 = \mathcal{R}_F^+ \iff PH^1(C^\circ_a)$

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<tr>
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Weight filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ is given by the following.

- Let $l_j$ ($1 \leq j \leq 3$) be the subspace of $\mathcal{R}_F$ spanned by the images of monomials $t_0^k t^m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $l_4 := \mathcal{R}_F$.

$$
\begin{array}{c}
\bullet l_1 \\
\bullet l_2 \\
\bullet l_3 \\
\bullet l_4 \\
\end{array}
$$

- $l_3 = \mathcal{R}_F^+ \iff PH^1(C_a^\circ)$

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## Summary of MHS

<table>
<thead>
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<td>$\mathbb{C} t_0$</td>
<td>$\mathbb{C} t_0^2$</td>
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<tr>
<td>$W_3/W_2$</td>
<td>$R_F^1/\mathbb{C} t_0$</td>
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</tbody>
</table>
| $W_4/W_3$ | $\mathbb{C} 1$ | $\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}$ | (1, 0)-form on $C_a$
| | | | (0, 1)-form on $C_a$
| | | | (1, 0)-form on $C_a$ with poles at $C_a - C_a^\circ$
## Summary of MHS

<table>
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<tr>
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<td>$W_4/W_3$</td>
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- (1, 0)-form on $C_a$
- (0, 1)-form on $C_a$
- (1, 0)-form on $C_a$ with poles at $C_a - C_a^o$

\[ \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \]
### Summary of MHS

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- $(1, 0)$-form on $C_a$
- $(0, 1)$-form on $C_a$
- $(1, 0)$-form on $C_a$ with poles at $C_a - C_a^0$

\[
\text{C}t_0, \quad \text{C}t_0^2, \quad R^1_F/\text{C}t_0
\]

\[
\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}
\]
## Summary of MHS

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- **(1, 0)-form on** $C_a$
- **(0, 1)-form on** $C_a$
- **(1, 0)-form on** $C_a$ with poles at $C_a - C_a^o$
### Summary of MHS

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- $\mathbb{C}t_0^2$ arrowed with $(0, 1)$-form on $C_a$
- $R_F^1/\mathbb{C}t_0$ arrowed with $(1, 0)$-form on $C_a$
- $\mathbb{C}1$ arrowed with $(1, 0)$-form on $C_a$ with poles at $C_a - C_a^0$
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$R_F^{1}/\mathbb{C}t_0$ → (1, 0)-form on $C_a$

(0, 1)-form on $C_a$

(1, 0)-form on $C_a$ with poles at $C_a - C_a^o$
# Summary of MHS

<table>
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- (1, 0)-form on $C_a$
- (0, 1)-form on $C_a$
- $R_{F_0}^1 / \mathbb{C} t_0$ with poles at $C_a - C_a^\circ$
Summary of MHS

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$(1, 0)$-form on $C_a$

$(0, 1)$-form on $C_a$

$(1, 0)$-form on $C_a$ with poles at $C_a - C_a^\circ$
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$(1, 0)$-form on $C_a$

$(0, 1)$-form on $C_a$

$(1, 0)$-form on $C_a$ with poles at $C_a - C_a^\circ$
Gauss–Manin connection

- So far, the parameter $a$ of $C_a$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.
- The Gauss–Manin connection $\nabla$ corresponds to the differential operators on $\mathcal{R}_F \otimes \mathbb{C}(a)$:

$\nabla \partial_{am} =: \nabla a_m \iff D_{am} := \partial_{am} + t_0 t^m \quad (m \in \Delta)$

<table>
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<td>$W_2/W_1$</td>
<td>$\nabla a_0$</td>
<td>$\mathbb{C}t_0$</td>
<td>$\nabla a_m$</td>
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<tr>
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<td></td>
<td></td>
<td>$\nabla a_m \ (m \neq 0)$</td>
</tr>
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Gauss–Manin connection

- So far, the parameter $a$ of $C_a$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.
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$$\nabla \partial_{a_m} =: \nabla_{a_m} \iff D_{a_m} := \partial_{a_m} + t_0 t^m \quad (m \in \Delta)$$

<table>
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<tr>
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<tr>
<td>$W_1$</td>
<td>$\mathbb{C}t_0$ \xrightarrow{am} $\mathbb{C}t_0^2$</td>
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<td>$W_2 / W_1$</td>
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$\nabla a_0 \rightarrow \mathbb{C} t_0 \quad \nabla_{a_m} \rightarrow \mathbb{C} t^2_0 \quad \nabla a_m (m \neq 0)$
Gauss–Manin connection

- So far, the parameter $a$ of $C^a_t$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.
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<td>$\mathbb{C}$</td>
<td>$\nabla a_0$</td>
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Remarks

- By the GM connection $\nabla$,
  - The Weight filtration is preserved: $\nabla_{a_m} W_k \subset W_k$
  - The Hodge filtration is changed by 1: $\nabla_{a_m} F^k \subset F^{k+1}$ (Griffiths transversality)

- It is easy to see that $\omega = (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in H^2(\mathbb{T}^2, C^\circ_a)$ satisfies the A-hypergeometric system with the parameter $\tilde{0}$. 
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  - The Hodge filtration is changed by 1: $\nabla_{a_m} F^k \subset F^{k+1}$ (Griffiths transversality)

- It is easy to see that $\omega = \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \in H^2(\mathbb{T}^2, C_a^\circ)$ satisfies the A-hypergeometric system with the parameter $\tilde{\theta}$. 
Yukawa coupling

In the case of $H^3(X^\vee)$ of a Calabi–Yau threefold $X^\vee$, the Yukawa coupling is

$$\int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega =: C_{ijk}.$$ 

In this definition, the polarization

$$H^3(X^\vee) \times H^3(X^\vee) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{X^\vee} \alpha \wedge \beta$$

is necessary.

In the case of $H^2(\mathbb{T}^2, C_a^\circ)$, we note that

$$W_1 H^2(\mathbb{T}^2, C_a^\circ) = H^1(C_a)$$

and use the polarization on $H^1(C_a)$ instead.
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$$W_1 H^2(\mathbb{T}^2, \mathcal{C}_a^0) = H^1(\mathcal{C}_a)$$

and use the polarization on $H^1(\mathcal{C}_a)$ instead.
**Definition**

Recall:

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<th>$H^1(C_a) = W_1$</th>
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<tr>
<td>$W_4/W_3$</td>
<td>$\mathbb{C}^1$</td>
<td>$\nabla a_0 R^{1}_{F/\mathbb{C}t_0}$</td>
<td>$(1,0)$-form on $C_a$</td>
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$\omega = \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \in H^2(\mathbb{T}^2, C_a^0)$

Therefore $\int_{C_a} \nabla^2 a_0 \omega \wedge \nabla a_0 \omega$ is well-defined.

We define this as the Yukawa coupling $\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle$.
Definition

Recall:

$H^1(C_a) = W_1$

$W_2/W_1$

$W_3/W_2$

$W_4/W_3$

$H^1(C_a)$

$F^2$

$F^1/F^2$

$F^0/F^1$

$(1, 0)$-form on $C_a$

$(0, 1)$-form on $C_a$

$\nabla a_0$

$\nabla a_0$

$\nabla t_0$

$\nabla t_0^2$

$R_F/\mathbb{C}t_0$

$\omega = (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in H^2(\mathbb{T}^2, C_a^0)$

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Recall:

\[
\begin{array}{|c|c|c|}
\hline
 & F^2 & F^1/F^2 & F^0/F^1 \\
\hline
H^1(C_a) = W_1 & C & C & C \\
W_2/W_1 & R^1_F/C_t_0 & C & C \\
W_3/W_2 & C & C & C \\
W_4/W_3 & C & C & C \\
\hline
\end{array}
\]

\[\nabla a_0 \phi_0 \rightarrow C_t_0 \rightarrow C_t_0^2 \rightarrow (1,0)\text{-form on } C_a\]

\[\nabla a_0 \phi_0 \rightarrow (0,1)\text{-form on } C_a\]

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\[ W_2/W_1 \]

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\[ W_4/W_3 \]

\[ F^2 \quad F^1/F^2 \quad F^0/F^1 \]

\[ \nabla a_0 \quad \nabla a_0 \quad \nabla a_0 \quad \nabla a_0 \quad \nabla a_0 \]

\[ \mathbb{C} t_0 \quad \mathbb{C} t_0^2 \]

\[ R^1_F/\mathbb{C} t_0 \]

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Definition

Recall:

\[
\begin{array}{|c|c|c|}
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\hline
H^1(C_a) = W_1 & & \\
W_2 / W_1 & & \\
W_3 / W_2 & R^{1}_F / \mathbb{C}t_0 & \\
W_4 / W_3 & \mathbb{C}1 & \\
\hline
\end{array}
\]

Therefore \[ \int_{C_a} \nabla^2_{a_0} \omega \wedge \nabla_{a_0} \omega \] is well-defined.

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Definition

Recall:

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W_2/W_1 & & \\
W_3/W_2 & & \\
W_4/W_3 & \mathbb{C}1 & \\
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\[\nabla_{a_0} \mathbb{C}t_0 \quad R^1_F/\mathbb{C}t_0 \quad (1, 0)\text{-form on } C_a \]

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We define this as the Yukawa coupling \[\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle\]
This can be generalized to other vector fields as follows.

- $\mathbb{L}$: the base space of the family (space of the parameter $a_m$'s)
- $T^0\mathbb{L}$: the subbundle of $T\mathbb{L}$ spanned by $\partial_{a_0}$

The Yukawa coupling is a multilinear map:

$$T\mathbb{L} \times T\mathbb{L} \times T^0\mathbb{L} \rightarrow \mathcal{O}_\mathbb{L},$$

$$\langle A, B; C \rangle := \int_{C_a} (\nabla_A \nabla_B \omega)' \wedge \nabla_C \omega$$

- $\nabla_C \omega \in F^1 \cap W_1$ is a $(1, 0)$-form on $C_a$
- $\nabla_A \nabla_B \omega$ may be outside of $W_1$. But such a class can be written as

$$\nabla_A \nabla_B \omega = \alpha_1 \quad (1, 0)\text{-form on } C_a \text{ with poles}$$
$$+ \alpha_2 \quad (0, 1)\text{-form on } C_a \text{ (without poles)}$$

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\[
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+ \alpha_2 \quad \text{(0, 1)-form on \( C_a \) (without poles)}
\]

So set \((\nabla_A \nabla_B \omega)' := \alpha_2\).
Remarks

- We can compute the Yukawa coupling by solving differential equations coming from A-hypergeometric system.

- Essentially, only $\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle$ is relevant:

$$
\langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = f_{ij}(a) \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle,
$$

where $\nabla_{a_i} \nabla_{a_j} \omega = t_0^2 t^{i+j} = f_{ij}(a)t_0^2 + (t_0\text{-degree} \leq 1)$.

Ex. $\Delta = \square$

$$
\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle = \frac{\text{const}}{a_0^3(1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})
$$

$$
\langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = \begin{cases} 
\frac{9a_0^2}{a_i a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i, j \neq 0) \\
\frac{3a_0}{a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (j = 0) 
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$$
\langle \partial a_i, \partial a_j; \partial a_0 \rangle = f_{ij}(a) \langle \partial a_0, \partial a_0; \partial a_0 \rangle,
$$

where $\nabla_m \nabla_m \omega = t_0^2 t^{i+j} = f_{ij}(a) t_0^2 + (t_0\text{-degree} \leq 1)$.

Ex.

$$
\Delta = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\quad
\langle \partial a_0, \partial a_0; \partial a_0 \rangle = \frac{\text{const}}{a_0^3 (1 + 27 z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})
$$

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where $\nabla_a \nabla_a \omega = t_0^2 t^{i+j} = f_{ij}(a) t_0^2 + (t_0\text{-degree} \leq 1)$.

Ex.

\[
\Delta = \begin{array}{c}
\begin{array}{c}
\text{const} \\
\frac{a_3}{a_0 (1 + 27z)}
\end{array}
\end{array}
\]

$$\langle \partial a_0, \partial a_0 ; \partial a_0 \rangle = \frac{\text{const}}{a_0^3 (1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})$$

$$\langle \partial a_i, \partial a_j ; \partial a_0 \rangle = \begin{cases} 
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Ex.

$$\Delta = \begin{array}{c}
\begin{array}{c}
\text{const} \\
\frac{a_1 a_2 a_3}{a_0^3(1 + 27z)} \\
(z = \frac{a_1 a_2 a_3}{a_0^3})
\end{array}
\end{array}$$

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\text{const} \\
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Ex. $\Delta = \begin{array}{c} \framebox{3} \\
\framebox{3} \end{array}$

$$\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle = \frac{\text{const}}{a_0^3(1+27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})$$

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Ex. $\Delta = \triangle$

$$\langle \partial a_0, \partial a_0; \partial a_0 \rangle = \frac{\text{const}}{a_0^3(1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})$$

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\frac{9a_i^2}{a_i a_j} \langle \partial a_0, \partial a_0; \partial a_0 \rangle & (i, j \neq 0) \\
\frac{3a_0}{a_j} \langle \partial a_0, \partial a_0; \partial a_0 \rangle & (j = 0)
\end{cases}$$
Yukawa coupling for the quotient family

- Consider the $\mathbb{T}^3$-action

$$\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2).$$

This is the action on the parameter space $\mathbb{L}$ and the family of curves $\sim$ the quotient family.

- The above definition of the Yukawa coupling is also valid for the quotient family.

Ex. $\Delta = \begin{diag}$

$$\mathcal{M} = \mathbb{L}/\mathbb{T}^3 \cong \mathbb{P}(1, 3) \setminus \{0, \frac{1}{27}\}. A \text{ local coordinate (around 0)} \text{ is } z = \frac{a_1 a_2 a_3}{a_0^3}.$$ 

$$\langle \partial_z, \partial_z; \partial_z \rangle = \frac{\text{const}}{27z^3(1 + 27z)}.$$ 

(Same as the known result)
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Ex. \[\Delta = \triangle\]

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**Ex.** $\Delta = \bigtriangleup$

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$$(\partial_z, \partial_z; \partial_z) = \frac{\text{const}}{27z^3(1 + 27z)}.$$
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- Consider the $\mathbb{T}^3$-action

$$\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2).$$

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(Same as the known result)
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\[ \mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2). \]

This is the action on the parameter space $\mathcal{L}$ and the family of curves $\sim$ the quotient family.

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**Ex.** \[ \Delta = \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \]

\[ \mathcal{M} = \mathcal{L}/\mathbb{T}^3 \cong \mathbb{P}(1, 3) \setminus \{0, \frac{1}{27}\}. \] A local coordinate (around 0) is $z = \frac{a_1 a_2 a_3}{a_0^3}$.

\[ \langle \partial_z, \partial_z; \partial_z \rangle = \frac{\text{const}}{27z^3(1 + 27z)}. \]

(Same as the known result)
Holomorphic anomaly eq.

In the B-model of mirror symmetry, there is BCOV’s holomorphic anomaly equation. It is a system of differential equations for higher genus prepotentials $F_g \ (g \geq 1)$. Let $\mathcal{M}$ be the complex moduli space of a Calabi–Yau 3-fold $X^\vee$, and $z_1, \ldots, z_n$ be its local coordinates. Holomorphic anomaly eq. involves:

- Kähler potential of $\mathcal{M}$: $K = -\log \sqrt{-1} \int_{X^\vee} \Omega \wedge \overline{\Omega}$
- Kähler metric on $\mathcal{M}$: $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$
- Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes^3)$

\[
\overline{\partial}_i F_g = \frac{1}{2} \sum_{j,k,j,k} \overline{C}_{ijk} e^{2K} G^{\bar{j}\bar{k}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})
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\[
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- Kähler metric on $\mathcal{M}$: $G_{i\bar{j}} = \partial_i \overline{\partial}_j K$
- Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, TM \otimes^3)$

$$\overline{\partial}_i F_g = \frac{1}{2} \sum_{j,k,i,j,k} \overline{C}_{ijk} e^{2K} G^{i\bar{j}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})$$
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\overline{\partial}_i F_g = \frac{1}{2} \sum_{j, k, \bar{j}, \bar{k}} C_{ijk} e^{2K} G^{j\bar{j}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})
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\[
\overline{\partial_i} F_g = \frac{1}{2} \sum_{j,k,j,k} \overline{C}_{ij\bar{k}} e^{2K} G^{j\bar{i}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})
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$$\overline{\partial}_i F_g = \frac{1}{2} \sum_{j,k,\bar{j},\bar{k}} C_{i\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})$$
Holomorphic anomaly eq. for Local B-model

For local B-model, we propose the following. We consider the quotient family of curves by the $\mathbb{T}^3$-action:

$$\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2).$$

Let $\mathcal{M} = \mathbb{L}/\mathbb{T}^3$, $z_1, z_2, \ldots$ be local coordinates of $\mathcal{M}$.

- $T^0 \mathcal{M} \subset T \mathcal{M}$: subbundle spanned by the image of $a_0 \partial_{a_0} =: \theta$.
- Hermitian metric on $T^0 \mathcal{M}$:

\[
(\theta, \theta) = \int_{C_a} \nabla_\theta \omega \wedge \overline{\nabla_\theta \omega} =: G_{0\bar{0}}
\]

($\partial_i := \partial_{z_i}$)

\[
\bar{\partial}_i F_0^{(g)} = \frac{\langle \partial_i, \theta; \theta \rangle}{2 G_{0\bar{0}}^2} (F_2^{(g-1)} + \sum_{r=1}^{g-1} F_1^{(r)} F_1^{(g-r)})
\]

\[
F_n^{(g+1)} := (\theta - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}}) F_n^{(g)}
\]
Holomorphic anomaly eq. for Local B-model

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$$

$$
F_{n+1}^{(g)} := (\theta - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}}) F_n^{(g)}
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\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2)
$$

Let $\mathcal{M} = \mathbb{L}/\mathbb{T}^3$, $z_1, z_2, \ldots$ be local coordinates of $\mathcal{M}$.

- $T^0\mathcal{M} \subset \mathcal{T}\mathcal{M}$: subbundle spanned by the image of $a_0 \partial a_0 =: \theta$.
- Hermitian metric on $T^0\mathcal{M}$:
  
  $$(\theta, \theta) = \int_{C_a} \nabla_\theta \omega \wedge \overline{\nabla_\theta \omega} =: G_{0\bar{0}}$$

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\partial_i F_0^{(g)} = \frac{\langle \partial_i, \theta; \theta \rangle}{2 G_{0\bar{0}}^2} (F_2^{(g-1)} + \sum_{r=1}^{g-1} F_1^{(r)} F_1^{(g-r)})
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Let $\mathcal{M} = \mathbb{L}/\mathbb{T}^3$, $z_1, z_2, \ldots$ be local coordinates of $\mathcal{M}$.

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($\partial_i := \partial_{z_i}$)

$$\overline{\partial_i} F^{(g)}_0 = \frac{\langle \partial_i, \theta; \theta \rangle}{2 G_{0\bar{0}}^2} \left( F^{(g-1)}_2 + \sum_{r=1}^{g-1} F^{(r)}_1 F^{(g-r)}_1 \right)$$

$$F^{(g)}_{n+1} := (\theta - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}}) F^{(g)}_n$$
Holomorphic anomaly eq. for Local B-model

For local B-model, we propose the following. We consider the quotient family of curves by the $T^3$-action:

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$$(\theta, \theta) = \int_{C_a} \nabla_\theta \omega \wedge \overline{\nabla_\theta \omega} =: G_{0\bar{0}}$$

$(\partial_i := \partial_{z_i})$

$$\overline{\partial_i} F_0^{(g)} = \frac{\langle \partial_i, \theta; \theta \rangle}{2 G_{0\bar{0}}^2} (F_2^{(g-1)} + \sum_{r=1}^{g-1} F_1^{(r)} F_1^{(g-r)})$$

$$F_{n+1}^{(g)} := (\theta - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}}) F_n^{(g)}$$
Remarks

- This holo. anomaly eq. is consistent with the following observations made previously by several authors [Klemm–Zaslow, Hosono, Haghhiat–Klemm–Rauch, Aganagic–Bouchard–Klemm, Alim–Länge–Mayr].

- needs no Kähler potential;

- Only the one dimensional subbundle $T^0 M$ of $TM$ matters. Similar to the one-parameter model.

- This can be solved by using the Feynman diagrams (with only one propagator) and also by Yamaguchi–Yau’s method.
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