Local B-model and Mixed Hodge Structure

Yukiko Konishi

1Department of Mathematics
Kyoto University

Mirror Symmetry and Gromov–Witten theory
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Local Mirror Symmetry

- Local mirror symmetry was derived by Chiang–Klemm–Yau–Zaslow in 1999.
- It is a statement about local GW invariants of smooth complete weak Fano toric surfaces.
- It is derived from mirror symmetry of toric Calabi–Yau hypersurfaces by considering a certain limits in moduli spaces.
  (e.g. CY hypersurface $\subset \hat{\mathbb{P}}(1,1,1,6,9) \sim \mathbb{P}^2$)

Local mirror symmetry is summarized as follows.
Local Mirror Symmetry

Start from:

\[ \Delta \]

2 dim reflexive polyhedron
Reflexive Polyhedra

A reflexive polyhedron $\Delta$ is a polyhedron satisfying:

- it is a convex hull of integral points;
- $0 \in \Delta$;
- Distance between 0 and each codimension 1 face is 1.

There are 16 2-dimensional reflexive polyhedra.
Local Mirror Symmetry

Δ
2 dim reflexive polyhedron

compact toric surface

\( \mathbb{P} \) s.t. \(-K_\mathbb{P}\) nef
• Regard integral points of $\Delta$ other than the origin as one dimensional cones in $\mathbb{R}^2$.
• Then they define a complete smooth 2-dimensional fan.
• This fan defines a complete smooth toric surface $\mathbb{P}$ whose anti-canonical divisor is nef.

Example

$\Delta \sim \mathbb{P}$
Local Mirror Symmetry

\[ \Delta \]

2 dim reflexive polyhedron

compact toric surface

\[ \mathbb{P} \text{ s.t. } -K_\mathbb{P} \text{ nef} \]

\[ g = 0 \text{ local GW inv.} \]

local A-model
Local Gromov–Witten invariants

Genus $g$ local Gromov–Witten invariant $N_{g,\beta}(\mathbb{P})$ of degree $\beta \in H_2(\mathbb{P}, \mathbb{Z})$ is defined by

$$N_{g,\beta}(\mathbb{P}) = \int_{[\overline{M}_{g,0}(\mathbb{P}, \beta)]^{vir}} e(\pi_* \ev^* K_{\mathbb{P}}).$$

- $\overline{M}_{g,n}(\mathbb{P}, \beta)$ is the moduli stack of stable maps to $\mathbb{P}$ of genus $g$ and degree $\beta$,
- $\ev : \overline{M}_{g,1}(\mathbb{P}, \beta) \to \mathbb{P}$ is the evaluation map,
- $\pi : \overline{M}_{g,1}(\mathbb{P}, \beta) \to \overline{M}_{g,0}(\mathbb{P}, \beta)$ is the map forgetting the marked point,
- $e$ denotes the Euler class.
- **Remark.** This is defined for $\beta$ such that $\beta \cdot K_{\mathbb{P}} \neq 0$. 
Local Mirror Symmetry

$\Delta$
2 dim reflexive polyhedron

compact toric surface
$\mathbb{P}$ s.t. $-K_\mathbb{P}$ nef
$g = 0$ local GW inv.
local A-model

a family of affine curves
$C_a \subset \mathbb{T}^2$
\[ \Delta \quad \sim \quad C_a^\circ \]

- Associate the Laurent monomial to an integral point in \( \Delta \):
  \[
  (m_1, m_2) \leftrightarrow t^m := t_1^{m_1} t_2^{m_2}
  \]
- Take the sum of these monomials with parameters \( a_m \):
  \[
  F_a(t) := \sum_{m \in \Delta} a_m t^m
  \]

Ex. \( \Delta = \)
\[
F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}.
\]

- The zero set of \( F_a(t) \) is an affine curve in \( \mathbb{T}^2 \):
  \[
  C_a^\circ = \{(t_1, t_2) \in \mathbb{T}^2 \mid F_a(t) = 0\}.
  \]
- \( C_a^\circ = \) genus 1 complete curve \(-\) points

Remark: we must put the \( \Delta \)-regularity condition on \( a_m \) so that \( C_a^\circ \) and its compactification are smooth.
Local Mirror Symmetry

local A,B-models are related to each other via

A-hypergeometric system with $\beta = \bar{0}$

$\Delta$

2 dim reflexive polyhedron

compact toric surface

$\mathbb{P}$ s.t. $-K_{\mathbb{P}}$ nef

$g = 0$ local GW inv.

local A-model

a family of affine curves

$C_{\alpha} \subset \mathbb{T}^2$

VMHS on $H^2(\mathbb{T}^2, C_{\alpha})$

local B-model

(a system of diff. eqs introduced by Gel’fand–Kapranov–Zelevinsky)

Studied by Stienstra, Batyrev
Local Mirror Symmetry

How are they related?

Solutions give:
- Mirror map
- A derivative of prepotential

Δ
2 dim reflexive polyhedron

A-hypergeometric system with $\beta = 0$

compact toric surface
$\mathbb{P}$ s.t. $-K_{\mathbb{P}}$ nef
$g = 0$ local GW inv.
local A-model

a family of affine curves
$C_a \subset \mathbb{T}^2$
VMHS on $H^2(\mathbb{T}^2, C_a)$
local B-model

PF equation for period integrals of "top element"
$(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$
Why $H^2(\mathbb{T}^2, C_a^\circ)$, Not $H^1(C_a^\circ)$?

1. It has a structure similar to $H^3$ of a Calabi–Yau 3-fold:

- **VHS on $H^3$**
  - Hodge filtration
    \[ 0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3 \]
    has dim $F^3 = 1$
  - $H^3$ generated by holo. 3-form $\in F^3$
  - GM connection

- **VMHS on $H^2(\mathbb{T}^2, C_a^\circ)$**
  - Hodge filtration
    \[ 0 \subset F^2 \subset F^1 \subset F^0 = H^2 \]
    has dim $F^2 = 1$
  - $H^2$ generated by
    \[ (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in F^2 \]
    and GM connection

2. Period integrals of $(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$ satisfy the
A-hypergeometric system.

- $H^1(C_a^\circ)$ does not have these properties!
Comparison with Mirror Symmetry

**MS**

- GW inv of $X$

**LMS**

- local GW inv of $\mathbb{P}$

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**A**

- VHS on $H^3(X^\vee)$
  - holo. 3-form $\Omega$
  - Yukawa coupling

\[ \int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega \]

**B**

- VMHS on $H^2(T^2, C_a)$
  - $\omega := (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)$
  - ??

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Important because it is:

- a third derivative of prepotential;
- necessary for BCOV’s holomorphic anomaly eq.
Our aim

- In several examples of local B-model, the Yukawa couplings have been computed [Klemm–Zaslow, Jinzenji–Forbes, Aganagic–Bouchard–Klemm, Haghihat–Klemm–Rauch, Alim–Länge-Mayr, Brini–Tanzini]. However, there has been no direct definition.

- We gave a definition of local B-model Yukawa coupling using the results of Batyrev, Stienstra on the VMHS on $H^2(T^2, C_a^\circ)$. 
Plan

- Mixed Hodge Structure of $H^2(\mathbb{T}^2, C^o_a)$ [Batyrev, Stientsra]
- Yukawa coupling
- Holomorphic anomaly equation
Mixed Hodge Structure of
\( H^2(\mathbb{T}^2, C^\circ_a) \)

- The mixed Hodge structure on \( H^2(\mathbb{T}^2, C^\circ_a) \) was studied by Batyrev (’93) and Stienstra (’97).
- \( H^2(\mathbb{T}^2, C^\circ_a) \) has a Jacobian-ring like description. It is isomorphic to a (quotient) vector space \( \mathcal{R}_F \), which is determined by the data of \( \Delta \) and \( F_a(t) \).
- The variation of Mixed Hodge structures on \( H^2(\mathbb{T}^2, C^\circ_a) \) is also described in terms of \( \mathcal{R}_F \).
- $\Delta(k)$: the polyhedron obtained by enlarging $\Delta$ by $k$-times.
- $\Delta(k)$: the polyhedron obtained by enlarging $\Delta$ by $k$-times.

\[
S^k_{\Delta} := \bigoplus_{m \in \Delta(k)} C t^k_0 t^m \quad (t^m := t^{m_1}_1 t^{m_2}_2)
\]

\[
S_\Delta := \bigoplus_{k \geq 0} S^k_\Delta, \quad \text{deg } t^k_0 t^m := k \quad \text{(a graded ring)}
\]

- Recall the defining equation $F_a(t)$ of $C^o_a$:

\[
F_a(t) = \sum_{m \in \Delta} a_m t^m.
\]

- Define the differential operators on $S_\Delta$: ($\theta_x := x \partial_x$)

\[
\mathcal{D}_0(t^k_0 t^m) = (k + t_0 F_a(t)) t^k_0 t^m
\]

\[
\mathcal{D}_i(t^k_0 t^m) = (m_i + t_0 \theta_t F_a(t)) t^k_0 t^m \quad (i = 1, 2).
\]

\[
\mathcal{R}_F := S_\Delta / \left( \sum_{i=0}^{2} \mathcal{D}_i S_\Delta \right)
\]
Example:

\[ \Delta = \triangle \]

- \( F_a(t) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} \)
- Relations \( \mathcal{D}_i1 = 0 \ (i = 0, 1, 2) \) imply:

\[
\begin{align*}
    t_0 t_1 &= -\frac{a_0}{3a_1} t_0, \\
    t_0 t_2 &= -\frac{a_0}{3a_2} t_0, \\
    \frac{t_0}{t_1 t_2} &= -\frac{a_0}{3a_3} t_0.
\end{align*}
\]

- By similar calculation, an element in \( S^k_\Delta \ (k \geq 2) \) is equal to

\[
\text{const.} t_0^2 + \text{an element in } S^1_\Delta.
\]

\[
\therefore \mathcal{R}_F \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2.
\]

For a reflexive polyhedron \( \Delta \),

\[
\mathcal{R}_F \cong \mathbb{C}1 \oplus R^1_F \oplus \mathbb{C}t_0
\]

\[
R^1_F := S^1_\Delta / \mathbb{C}t_0 F_a \oplus \mathbb{C}t_1 \theta t_1 F_a \oplus \mathbb{C}\theta t_2 F_a
\]
\[ \mathcal{R}_F \cong H^2(\mathbb{T}^2, C^\circ_a) \]

Stienstra, Batyrev showed that \( \mathcal{R}_F \cong H^2(\mathbb{T}^2, C^\circ_a) \). This isomorphism is as follows.

- Note that
  \[ \mathcal{R}_F \cong \mathcal{R}_F^+ \oplus \mathbb{C}1 \]
  \( (\mathcal{R}_F^+ \subset \mathcal{R}_F : \text{spanned by monomials } t_0^k t_m \text{ with } k \geq 1) \)

- There is an exact sequence
  \[ 0 \longrightarrow PH^1(C^\circ_a) \longrightarrow H^2(\mathbb{T}^2, C^\circ_a) \longrightarrow H^2(\mathbb{T}^2) \longrightarrow 0 \]
  \( (PH^1(C^\circ_a) := H^1(C^\circ_a)/H^1(\mathbb{T}^2)) \)

\[ \mathbb{C}1 \leftrightarrow H^2(\mathbb{T}^2)\text{-part} : \quad 1 \leftrightarrow \left( \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0 \right) \]

\[ \mathcal{R}_F^+ \leftrightarrow PH^1(C^\circ_a)\text{-part} : \]

\[ t_0^k t_m \leftrightarrow (0, \text{Res}_{F_a=0} \frac{(-1)^{k-1}(k-1)!t^m}{F^k_a} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}) \]
Example

\[ \Delta = \begin{array}{c}
\end{array} \]

- \[ \mathcal{R}_F \cong \frac{\mathbb{C}^1}{H^2(\mathbb{T}^2)} \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \]

- \[ PH^1(C_a^\circ) = H^1(C_a) \quad (C_a \text{ is a compactification of } C_a^\circ) \]

\[ 1 \iff \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \text{ on } \mathbb{T}^2 \]

\[ t_0 \iff (1, 0)-\text{form on } C_a \]

\[ t_0^2 \iff (0, 1)-\text{form on } C_a \]
What’s Mixed Hodge Structure?

- $H^k(V)$ of a smooth projective variety $V$ has the canonical Hodge structure of weight $k$:

$$H^{p,q} = H^{p,q}(V) \quad \text{(Hodge decomposition)}$$

- Mixed Hodge structure is, in a sense, a generalization of Hodge structure to $H^k(U)$ of an open variety $U$.

- Mixed Hodge structure of weight $k$ consists of:
  - free abelian group $H_\mathbb{Z}$,
  - the weight filtration $W_\bullet$ on $H_\mathbb{Z}$ (increasing filtration),
  - the Hodge filtration $F^\bullet$ on $H_\mathbb{C}$ (decreasing filtration),

such that the induced Hodge filtration on $W_l/W_{l-1}$ has a Hodge structure of weight $l + k$.

$$H^{p,k+1-l-p} := \frac{F^p W_l/W_{l-1}}{F^{p+1} W_l/W_{l-1}} \quad \text{satisfy } H^{p,q} = \overline{H}^{q,p}.$$
• If $U = V - D$ where $V$ is a smooth projective variety and $D$ is a simple normal crossing divisor, then $H^k(U)$ has a canonical mixed Hodge structure.

• Hodge filtration $F^\bullet$ is induced from the filtration on $\Omega^\bullet_V(\log D)$

$$F^p\Omega^\bullet_V(\log D) = \Omega^\geq p_V(\log D)$$

• Weight filtration is induced from the filtration

$$W_l\Omega^\bullet_V(\log D) = \wedge^l\Omega^1_V(\log D) \wedge \Omega^{\bullet - l}_V.$$  

Roughly speaking, $W_{k+l} \subset H^k(U)$ consists of forms on $V$ with logarithmic poles on $D$ of order at most $l$.

• For the relative cohomology of the pair $U_1 \subset U_2$, there is a canonical MHS. The long exact sequence

$$\ldots \longrightarrow H^k(U_1) \longrightarrow H^{k+1}(U_2, U_1) \longrightarrow H^{k+1}(U_2) \longrightarrow \ldots$$

is a long exact sequence of MHS’s.
Hodge filtration

Hodge filtration on $H^2(\mathbb{T}^2, C_a)$ is given by the filtration on $\mathcal{R}_F$:

- Let $\mathcal{E}^{-i}$ ($i = 0, 1, 2, \ldots$) be the subspace of $\mathcal{R}_F$ spanned by the images of all monomials of the $t_0$-degree $\leq i$.

Ex.

- $\mathcal{E}^0 = \mathbb{C}1 \iff H^2(\mathbb{T}^2), \mathcal{R}_F = \mathcal{E}^{-2} = \mathcal{E}^{-3} = \mathcal{E}^{-4} = \ldots$

\[
\begin{array}{c|c|c|c|c}
0 & \mathcal{E}^0 & \mathcal{E}^{-1} & \mathcal{E}^{-2} & \mathcal{R}_F \\
\hline
0 & \mathcal{F}^2 & \mathcal{F}^1 & \mathcal{F}^0 & H^2(\mathbb{T}^2, C_a)
\end{array}
\]
Weight filtration

Weight filtration on $H^2(\mathbb{T}^2, C_\alpha^\circ)$ is given by the following.

- Let $l_j (1 \leq j \leq 3)$ be the subspace of $\mathcal{R}_F$ spanned by the images of monomials $t_0^k t_1^m$ with $k \geq 1$ such that $m \in \Delta(k)$ does not belong to any face of codimension $j$. Set $l_4 := \mathcal{R}_F$.

- $l_3 = \mathcal{R}_F^+ \Leftrightarrow PH^1(C_\alpha^\circ)$

\[
\begin{array}{cccccc}
0 & \subset & l_1 & \subset & l_2 & \subset & l_3 & \subset & l_4 &= & \mathcal{R}_F \\
& \| & & \| & & \| & & \| & & \| & & \\
0 & \subset & W_1 & \subset & W_2 & = & W_3 & \subset & W_4 &= & H^2(\mathbb{T}^2, C_\alpha^\circ)
\end{array}
\]
## Summary of MHS

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<tr>
<th>$W_i$</th>
<th>$F^2$</th>
<th>$F^1/F^2$</th>
<th>$F^0/F^1$</th>
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<td>$W_1$</td>
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<td>$\mathbb{C}t_0^2$</td>
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<td>$W_2/W_1$</td>
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<td>$R_F^1/\mathbb{C}t_0$</td>
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<td>$W_3/W_2$</td>
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<td>$W_4/W_3$</td>
<td>$\mathbb{C}1$</td>
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**Gauss–Manin connection**

- So far, the parameter $a$ of $C_a$ is fixed. From now on, we move $a$ (in the range such that $F_a(t)$ is $\Delta$-regular) and consider the family of affine curves.
- The Gauss–Manin connection $\nabla$ corresponds to the differential operators on $\mathcal{R}_F \otimes \mathbb{C}(a)$:

\[
\nabla \partial_{a_m} =: \nabla a_m \Leftrightarrow D_{a_m} := \partial_{a_m} + t_0 t^m \quad (m \in \Delta)
\]

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- $\nabla a_0$ via $\mathbb{C} t_0$
- $\nabla a_m$ via $\mathbb{C} t_0^2$
- $\nabla a_m$ via $R^1_F/\mathbb{C} t_0$
- $\nabla a_m$ (for $m \neq 0$) via $\mathbb{C} 1$
By the GM connection $\nabla$,

- The Weight filtration is preserved: $\nabla a_m W_k \subset W_k$
- The Hodge filtration is changed by 1: $\nabla a_m F^k \subset F^{k+1}$ (Griffiths transversality)

It is easy to see that $\omega = (\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0) \in H^2(\mathbb{T}^2, C_a^\circ)$ satisfies the A-hypergeometric system with the parameter $\tilde{\theta}$. 

Remarks
Yukawa coupling

In the case of $H^3(X^\vee)$ of a Calabi–Yau threefold $X^\vee$, the Yukawa coupling is

$$\int_{X^\vee} \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega =: C_{ijk}.$$ 

In this definition, the polarization

$$H^3(X^\vee) \times H^3(X^\vee) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{X^\vee} \alpha \wedge \beta$$

is necessary.

In the case of $H^2(\mathbb{T}^2, C_a^\circ)$, we note that

$$W_1 H^2(\mathbb{T}^2, C_a^\circ) = H^1(C_a)$$

and use the polarization on $H^1(C_a)$ instead.
Recall:

\[ H^1(C_a) = W_1 \]
\[ W_2 / W_1 \]
\[ W_3 / W_2 \]
\[ W_4 / W_3 \]

\[ \mathbb{C}^1 \]

\[ \nabla_{a_0} \]
\[ R^1_F / \mathbb{C}t_0 \]
\[ \nabla_{a_0} \]
\[ \mathbb{C}t_0 \]
\[ \mathbb{C}t_0^2 \]

Therefore \( \int_{C_a} \nabla^2_{a_0} \omega \wedge \nabla_{a_0} \omega \) is well-defined.

We define this as the Yukawa coupling \( \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle \)
This can be generalized to other vector fields as follows.

- \( \mathbb{L} \): the base space of the family (space of the parameter \( a_m \)'s)
- \( T^0 \mathbb{L} \): the subbundle of \( T \mathbb{L} \) spanned by \( \partial_{a_0} \)

The Yukawa coupling is a multilinear map:

\[
T \mathbb{L} \times T \mathbb{L} \times T^0 \mathbb{L} \rightarrow \mathcal{O}_\mathbb{L},
\]

\[
\langle A, B; C \rangle := \int_{C_a} (\nabla_A \nabla_B \omega)' \wedge \nabla_C \omega
\]

- \( \nabla_C \omega \in F^1 \cap W_1 \) is a \( (1, 0) \)-form on \( C_a \)
- \( \nabla_A \nabla_B \omega \) may be outside of \( W_1 \). But such a class can be written as

\[
\nabla_A \nabla_B \omega = \alpha_1 \quad \text{(1, 0)-form on } C_a \text{ with poles}
\]

\[
+ \alpha_2 \quad \text{(0, 1)-form on } C_a \text{ (without poles)}
\]

So set \( (\nabla_A \nabla_B \omega)' := \alpha_2 \).
Remarks

- We can compute the Yukawa coupling by solving differential equations coming from A-hypergeometric system.

- Essentially, only $\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle$ is relevant:

$$\langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = f_{ij}(a) \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle,$$

where

$$\nabla_{a_i} \nabla_{a_j} \omega = t_0^2 t^{i+j} = f_{ij}(a)t_0^2 + (t_0\text{-degree} \leq 1).$$

Ex.

$$\Delta = \begin{array}{c}
\end{array}$$

$$\langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle = \frac{\text{const}}{a_0^3(1 + 27z)} \quad (z = \frac{a_1 a_2 a_3}{a_0^3})$$

$$\langle \partial_{a_i}, \partial_{a_j}; \partial_{a_0} \rangle = \begin{cases} 
\frac{9a_0}{a_i a_j} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i, j \neq 0) \\
\frac{3a_0}{a_i} \langle \partial_{a_0}, \partial_{a_0}; \partial_{a_0} \rangle & (i = 0)
\end{cases}$$
Yukawa coupling for the quotient family

- Consider the $\mathbb{T}^3$-action

$$\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2).$$

This is the action on the parameter space $\mathbb{L}$ and the family of curves $\sim \to$ the quotient family.

- The above definition of the Yukawa coupling is also valid for the quotient family.

**Ex.** $\Delta = \triangle$

$$\mathcal{M} = \mathbb{L}/\mathbb{T}^3 \cong \mathbb{P}(1, 3) \setminus \{0, \frac{1}{27}\}. A local coordinate (around 0) is \(z = \frac{a_1 a_2 a_3}{a_0^3}\).$$

$$\langle \partial_z, \partial_z; \partial_z \rangle = \frac{\text{const}}{27z^3(1 + 27z)}.$$
Holomorphic anomaly eq.

In the B-model of mirror symmetry, there is BCOV’s holomorphic anomaly equation. It is a system of differential equations for higher genus prepotentials $F_g \ (g \geq 1)$. Let $\mathcal{M}$ be the complex moduli space of a Calabi–Yau 3-fold $X^\vee$, and $z_1, \ldots, z_n$ be its local coordinates.

Holomorphic anomaly eq. involves:

- Kähler potential of $\mathcal{M}$: $K = - \log \sqrt{-1} \int_{X^\vee} \Omega \wedge \overline{\Omega}$
- Kähler metric on $\mathcal{M}$: $G_{i\bar{j}} = \partial_i \overline{\partial}_{\bar{j}} K$
- Yukawa coupling $C_{ijk} \in \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M} \otimes^3)$

\[
\overline{\partial}_j F_g = \frac{1}{2} \sum_{j,k,j,\bar{k}} \overline{C}_{i\bar{j}k} e^{2K} G^{\bar{j}j} G^{k\bar{k}} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r \cdot D_k F_{g-r})
\]
Holomorphic anomaly eq. for Local B-model

For local B-model, we propose the following.

We consider the quotient family of curves by the $\mathbb{T}^3$-action:

$$\mathbb{T}^3 \ni \lambda : F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2) .$$

Let $\mathcal{M} = \mathbb{L}/\mathbb{T}^3$, $z_1, z_2, \ldots$ be local coordinates of $\mathcal{M}$.

- $T^0 \mathcal{M} \subset T \mathcal{M}$: subbundle spanned by the image of $\partial_0 \partial_{\lambda_0} =: \theta$.
- Hermitian metric on $T^0 \mathcal{M}$:

$$(\theta, \theta) = \int_{\mathcal{C}_a} \nabla_\theta \omega \wedge \overline{\nabla_\theta \omega} =: G_{0\bar{0}}$$

$(\partial_i := \partial_{z_i})$

\[
\bar{\partial}_i F_0^{(g)} = \frac{\langle \bar{\partial}_i, \theta; \theta \rangle}{2 G_{0\bar{0}}^2} (F_2^{(g-1)} + \sum_{r=1}^{g-1} F_1^{(r)} F_1^{(g-r)})
\]

\[
F_0^{(g)} = (\theta - n \frac{\theta G_{0\bar{0}}}{G_{0\bar{0}}}) F_0^{(g)}
\]
Remarks

- This holo. anomaly eq. is consistent with the following observations made previously by several authors [Klemm–Zaslow, Hosono, Haghihat–Klemm–Rauch, Aganagic–Bouchard–Klemm, Alim–Länge–Mayr].

- needs no Kähler potential;
- Only the one dimensional subbundle $T^0M$ of $TM$ matters. Similar to the one-parameter model.

- This can be solved by using the Feynman diagrams (with only one propagator) and also by Yamaguchi–Yau’s method.