Linear grading function and further reduction of normal forms

Hiroshi Kokubu*
Department of Mathematics
Kyoto University
Kyoto 606-01, Japan

Hiroe Oka†
Department of Applied Mathematics and Informatics
Faculty of Science and Technology
Ryukoku University
Seta, Otsu 520-21, Japan
oka@rins.ryukoku.ac.jp

and

Duo Wang‡
Department of Applied Mathematics
Tsinghua University
Beijing 100084, P. R. China
dwang@ccastb.ccast.ac.cn

March 6, 1996

Abstract

In this note a kind of new grading functions is introduced, the definition of N-th order normal form is given and some sufficient conditions for the uniqueness of normal forms are derived. A special case of the unsolved problem in Baider and Sanders paper for the unique normal form of Bogdanov-Takens singularities is solved.

Keywords: grading function, normal form.

* Research was supported in part by Grant-in-Aid for Scientific Research (No. 07740150), Ministry of Education, Science and Culture, Japan.

† Research was supported in part by Science and Technology Fund for Research Grants of Ryukoku University.

‡ Research was supported in part by NNSF of China

1 Introduction

Normal forms are basic and powerful tools in bifurcation theory of vector fields. But classical normal form theory, known as Poincaré's normal form (see, e.g., Arnold [Ar]), may not give the simplest form since only linear parts are used for simplifying the nonlinear terms, and hence one can not apply Poincaré normal form theory to vector fields whose linear parts are identically zero. On the other hand, classical normal forms are not unique in general. In order to get unique normal forms so that formal classification could be made, further reduction of the classical normal forms is necessary and the concept of normal forms should be refined.

Many authors have discussed the further reduction of normal forms and some of them have discussed uniqueness of normal forms, see, e.g., [SM] and references therein. Ushiki [Us] introduced a systematic method by which nonlinear parts are also used to simplify higher order terms. By classical method of normal form theory, only one Lie bracket is used to simplify the higher order terms. Ushiki's method allows more Lie brackets for the simplification [CK]. They obtained unique normal forms (simplest normal forms) up to some degree for some given vector fields. Wang [Wa] gave a method to calculate coefficients of normal forms, which needs more parameters in the transformations due to the non-uniqueness of transformations and hence may give simplest normal forms (up to some finite order) by suitably choosing parameters. In fact nonlinear terms play also role in the reduction. Baider introduced special form [Ba], which is in fact unique normal form, in an abstract sense. Baider and Sanders [BS1] introduced new grading functions to get further reduction of normal forms. They introduced the concept of n-th order normal form related with the n-th grading function and give the definition of infinite order normal forms (which is unique). They gave unique normal forms for some nilpotent Hamiltonians. Then they got unique normal forms for some cases of Bogdanov-Takens singularities ([BS2]). But some cases are still unsolved. Results concerning uniqueness of normal forms for some other cases can be found in [BC2] and [SM].

In this paper we first introduce the concept of linear grading function in section 2, and we give a systematic method to define some new grading functions. Then in section 3 we define n-th order normal forms, in which we combine Ushiki's method and Baider and Sanders' method. In fact we need only one grading function. But the n-th order normal forms related to n Lie brackets in the computation. In section 4 we define infinite order normal form and prove that the infinite order normal form must be unique. In section 5 we give a sufficient condition for unique normal forms. Finally in section 6 we prove the uniqueness of first order normal form of the special case $\mu = 2, \nu = 1$ of Bogdanov-Takens singularities which is a special case of the unsolved problem in [BS2].

2 Linear grading function

Let H be the linear space of all n dimensional real or complex formal vector fields. We define a bilinear operator $[\cdot,\cdot] : H \times H \to H$ by $[u,v] = Du\cdot v - Dv\cdot u$ for any $u, v \in H$. Then $\{H, [\ ,\ ]\}$ forms a Lie algebra. Now let us define a "grading function" such that $\{H, [\ ,\ ]\}$ is a graded Lie algebra.

For the purpose of computing normal forms of formal vector fields, the "grading function" should satisfy the following properties:

(1) The degree of any monomial is defined to be an integer. The dimension of the linear space H_k spanned by all monomials of degree k is finite for any integer k (in the case when there is no monomials of degree k for some integer k we define $H_k = \{0\}$);

(2) $[H_m, H_n] \subset H_{m+n}$ for any integers m, n;

(3) The grading function should be bounded below.

Let

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} e_j \mid l_i \in \mathbb{Z}^+, x_i \in \mathbb{R} \text{ (or } \mathbb{C}), i, j = 1, ..., n \right\},$$

where e_j is the j-th standard unit vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Consider the function $\delta : D_n \to \mathbb{Z}$ defined by

$$\delta\left(\prod_{i=1}^{n} x_i^{l_i} e_j\right) = \sum_{i=1}^{n} a_{ij} l_i + d_j, \tag{1}$$

where $a_{ij},\ d_j \in \mathbb{Z},\ i, j = 1, ..., n$. From the definition of δ, it is obvious that condition 1. for a grading function is satisfied. Now we look for conditions such that the function δ defined by (1) satisfies all other conditions of grading functions.

Lemma 2.1 *The function δ defined by (1) is bounded below if and only if all $\{a_{ij}\}$ are nonnegative integers.*

Lemma 2.2 *Let the function δ be defined by (1) with all coefficients $\{a_{ij}\}$ and $\{d_j\}$ nonnegative and H_k be the linear space spanned by all monomials in $\delta^{-1}(k)$. Then $dim(H_k)$ is finite(or zero) for any integer k if and only if all $\{a_{ij}\}$ are natural numbers.*

Lemma 2.3 *Let the function δ be defined by (1) and H_k be defined as in Lemma 2.2. Then $[H_m, H_n] \subset H_{m+n}$ if and only if*

$$a_{i1} = ... = a_{in} = -d_i \textit{ for any } i = 1, ..., n.$$

Proof. Let $u = \prod_{i=1}^{n} x_i^{l_i} e_j, v = \prod_{i=1}^{n} x_i^{l'_i} e_k$ and $\delta(u) = m, \delta(v) = n$. Then

$$\begin{aligned} [u, v] = & \ Du \cdot v - Dv \cdot u \\ = & \ \tfrac{l_k}{x_k} \prod_{i=1}^{n} x_i^{l_i} \cdot - \tfrac{l'_j}{x_j} \prod_{i=1}^{n} x_i^{l'_i} \cdot \prod_{i=1}^{n} x_i^{l_i} e_k \\ = & \ \tfrac{l_k}{x_k} \prod_{i=1}^{n} x_i^{l_i+l'_i} e_j - \tfrac{l'_k}{x_k} \prod_{i=1}^{n} x_i^{l_i+l'_i} e_k. \end{aligned} \tag{2}$$

We first assume that $a_{i1} = a_{i2} = ... = a_{in} = -d_i, i, j = 1, \cdots, n$. Then

$$\begin{aligned} \delta\left(\tfrac{1}{x_k} \prod_{i=1}^{n} x_i^{l_i+l'_i} e_j\right) & = \sum_{i=1}^{n} a_{ij}(l_i + l'_i) - a_{kj} + d_j \\ & = \left(\sum_{i=1}^{n} a_{ij} l_i + d_j\right) + \left(\sum_{i=1}^{n} a_{ik} l'_i + d_k\right) = m + n, \end{aligned}$$

and

$$\begin{aligned} \delta\left(\tfrac{1}{x_j} \prod_{i=1}^{n} x_i^{l_i+l'_i} e_k\right) & = \sum_{i=1}^{n} a_{ik}(l_i + l'_i) - a_{jk} + d_k \\ & = \left(\sum_{i=1}^{n} a_{ik} l_i + d_k\right) + \left(\sum_{i=1}^{n} a_{ij} l'_i + d_j\right) = n + m. \end{aligned}$$

Hence, from(2), we have $[u, v] \in H_{m+n}$. Note that the operator $[\cdot, \cdot]$ is bilinear. Therefore $[H_m, H_n] \subset H_{m+n}$.

Conversely, we suppose that $[H_m, H_n] \subset H_{m+n}$ holds for any integer m, n. For any $k \in \mathbb{N}$, we fix a $u \in H_m$ with $l_k > 0$. Then from (2) we have

$$\sum_{i=1}^{n} a_{ij}(l_i + l'_i) - a_{kj} + d_j = \left(\sum_{i=1}^{n} a_{ij} l_i + d_j\right) + \left(\sum_{i=1}^{n} a_{ik} l'_i + d_k\right).$$

Hence

$$\sum_{i=1}^{n}(a_{ik}-a_{ij})l'_i + d_k + a_{kj} = 0,$$

i. e.

$$d_k + a_{kj} = \sum_{i=1}^{n}(a_{ij}-a_{ik})l'_k.$$

If we take $l'_i = \begin{cases} l, & i = k, \\ 0, & i \neq k, \end{cases}$ where $l \in \mathbb{N}$, then

$$a_{kj} - a_{kk} = \frac{d_k + a_{kj}}{l}. \tag{3}$$

Letting $l \to +\infty$, we have $a_{kj} = a_{kk}$. Note that j is arbitrary. Therefore $a_{k1} = ... = a_{kk} = ... = a_{kn}$, and hence from (3), $d_k = -a_{k1}$ follows.

Definition 2.4 *Let*

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} e_j \mid l_i \in \mathbb{Z}^+, x_i \in \mathbb{R} \ (or\ \mathbb{C}), i, j = 1, ..., n \right\},$$

where e_j is the j-th standard unit vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then the function $\delta : D_n \to \mathbb{Z}$ defined by

$$\delta\left(\prod_{i=1}^{n} x_i^{l_i} e_j\right) = \sum_{i=1}^{n} a_i l_i - a_j, \tag{4}$$

where $a_i, \in \mathbb{N}$, $i = 1, ..., n$, is called a linear grading function.

Remark 2.5 *1)Suppose that δ is a linear grading function defined by a set of natural numbers $\{a_i\}$. If $\{a_i\}$ has a common factor c, then the function $\frac{1}{c}\delta$ is also a linear grading function. So we will always assume that any linear grading function is defined by a set of coprime natural numbers $\{a_i\}$.*

2)Any linear grading function δ satisfies

$$\delta(x_i e_i) = 0 \quad \text{for all} \quad i = 1, ..., n.$$

Hence for any grading function δ, $\min_p \delta(p) \leq 0$.

3)If the linear grading function δ is defined by a set of successive natural numbers $\{a_1, ..., a_n\}$, then for $\forall k \geq 1-n$, $dim H_k \geq 1$.

Example 2.6 *If* $\delta\left(\prod_{i=1}^{n} x_i^{l_i} e_j\right) = \sum_{i=1}^{n} l_i - 1$, *i.e.* $a_1 = ... = a_n = 1$, *then* δ *is a linear grading function. Note that the classical definition of the degree of* $\prod_{i=1}^{n} x_i^{l_i} e_j$ *is* $\sum_{i+1}^{n} l_i$ *and hence the grading function defined above shifts by 1 w.r.t the classical grading.*

Example 2.7 *In* $D_2 = \left\{\prod_{i=1}^{2} x_i^{l_i} e_j,\ j = 1, 2\right\}$, *define*

$$\delta\left(x_1^{l_1} x_2^{l_2} e_j\right) = \begin{cases} 2l_1 + 3l_2 - 2, & j = 1, \\ 2l_1 + 3l_2 - 3, & j = 2. \end{cases}$$

Then $\delta(x_2 e_1) = \delta(x_1^2 e_2) = 1$. *Note that* $x_2 e_1$ *is a linear term and* $x_1^2 e_2$ *is a nonlinear term in the classical sense.*

3 N-th order normal forms

Let δ be a linear grading function and H_k be the linear space spanned by all monomials of degree k. Consider a formal vector field V defined by the following formal series

$$X = X_\mu + X_{\mu+1} + ... + X_{\mu+k} + ..., \tag{5}$$

where $X_k \in H_k,\ k \geq \mu$ and $X_\mu \neq 0$. We call (5) a zeroth order normal form and denote it as

$$V^{(0)} = V_\mu^{(0)} + X_{\mu+1} + ... + X_{\mu+k} + ... \tag{6}$$

We may assume that X_μ is already in some simple or satisfactory form(e.g. X_μ may have been changed to simpler form by classical normal theory).

Let $Y_k \in H_k$ and Φ_{Y_k} be its time one mapping given by the flow $\Phi_{Y_k}^t$ generated from the vector field corresponding to the equation $\dot{x} = Y_k(x),\ x \in \mathbb{R}^n$. Then the transformation $y = \Phi_{Y_k}(x)$, which is a near identity change of variables, brings (5) to

$$\begin{aligned} \Phi_{Y_k *} X &= \exp(adY_k) X \\ &= X + (adY_k) X + ... + \tfrac{1}{n!}(adY_k)^n X + ..., \end{aligned}$$

where $(adY_k)X = [Y_k, X]$ and $(adY_k)^n = (adY_k)^{n-1} \cdot (adY_k),\ n = 2, 3, ...$

For any $k \in \mathbb{N}$, define an operator

$$L_k^{(1)} : H_k \to H_{\mu+k} :\ Y_k \mapsto [Y_k, V_\mu^{(0)}]. \tag{7}$$

It is obvious that $L_k^{(1)}$ is linear. Note that $L_k^{(1)}$ depends on $V_\mu^{(0)}$ and can be denoted by $L_k^{(1)} = L_k^{(1)}[V_\mu^{(0)}]$.

Definition 3.1

$$V = V_\mu + V_{\mu+1} + ... + V_{\mu+k} + ...$$

is called a first order normal form, *if*

$$V_{\mu+k} \in N_{\mu+k}^{(1)},\ k = 1, 2, ...,$$

where $N_{\mu+k}^{(1)}$ is a complement subspace to $ImL_k^{(1)}$ in $H_{\mu+k}$ and $L_k^{(1)} = L_k^{(1)}[V_\mu]$.

It is easy to see that there is a sequence of near identity formal transformations such that (5) is transformed into a first order normal form which is called the first order normal form of (5) and can be denoted by

$$V^{(1)} = V_\mu^{(1)} + V_{\mu+1}^{(1)} + ... + V_{\mu+k}^{(1)} + ... \tag{8}$$

Note that $V_\mu^{(1)} = V_\mu^{(0)}$.

In order to make further reduction of a first order normal form, we define a sequence of linear operators $L_k^{(m)}, m, k = 1, 2, 3, \cdots$ as follows. Let

$$V = V_\mu + V_{\mu+1} + V_{\mu+2} + \cdots + V_{\mu+k} + \cdots$$

be a formal series, where $V_m \in H_m$ for each $m \geq \mu$. Then we define $L_k^{(1)} = L_k^{(1)}[V_\mu]$ by (7) for any $k \in \mathbb{N}$; if $L_k^{(m)} = L_k^{(m)}[V_\mu, V_{\mu+1}, \cdots, V_{\mu+m-1}]$ is defined already for an $m \geq 1$ and any $k \in \mathbb{N}$, then we define $L_k^{(m+1)} = L_k^{(m+1)}[V_\mu, V_{\mu+1}, \cdots, V_{\mu+m}]$ by

$$\begin{aligned} L_k^{(m+1)} :\quad & KerL_k^{(m)} \times H_{m+k} \to H_{\mu+m+k} : ((Y_k, Y_{k+1}, ..., Y_{k+m-1}), Y_{k+m}) \mapsto \\ & [Y_k, V_{\mu+m}] + ... + [Y_{k+m-1}, V_{\mu+1}] + [Y_{k+m}, V_\mu]. \end{aligned}$$

Remark 3.2 *By definition, it is obvious that*

$$
\begin{aligned}
KerL_k^{(m)} \quad = \quad & \{(Y_k, Y_{k+1}, ..., Y_{k+m-1}) \in H_k \times ... \times H_{k+m-1} \mid \\
& [Y_k, V_\mu] = 0, \\
& [Y_{k+1}, V_\mu] + [Y_k, V_{\mu+1}] = 0, \\
& \vdots \\
& [Y_{k+m-1}, V_\mu] + ... + [Y_k, V_{\mu+m-1}] = 0\}.
\end{aligned}
$$

Definition 3.3 *A formal vector field*

where $V_m \in H_m$ for each $m \geq \mu$, is called an N -th order normal form, *if*

$$V_{\mu+i} \in N_{\mu+i}^{(i)} \ (1 \leq i \leq N-1),$$

and

$$V_{\mu+j} \in N_{\mu+j}^{(N)} \ (j \geq N),$$

where $N_{\mu+k}^{(m)}$ is a complement to the image of $L_{k-m+1}^{(m)}[V_\mu, V_{\mu+1}, \cdots, V_{\mu+m-1}]$ in $H_{\mu+k}$ for each $m \geq 1$ and $k \geq 1$.

Theorem 3.4 *For any $N \in \mathbb{N}$, any formal vector field can be changed by a sequence of near identity formal transformations to an N-th order normal form.*

Proof. Consider a formal vector field(a zero-th order normal form)

$$V^{(0)} = V_\mu^{(0)} + X_{\mu+1} + ... + X_{\mu+k} + ... \tag{9}$$

Define linear operator $L_1^{(1)} = L_1^{(1)}[V_\mu^{(0)}]$ and let

$$H_{\mu+1} = ImL_1^{(1)} \oplus N_{\mu+1}^{(1)}.$$

Then there is a polynomial $Y^1 = Y_1^{(1)} \in H_1$ such that (9) is converted to

$$V^{(1)} = exp(adY^1)V^{(0)} = V_\mu^{(0)} + V_{\mu+1}^{(1)} + X_{\mu+2}^{(1)} + \cdots, \tag{10}$$

where $V^{(1)}_{\mu+1} \in N^{(1)}_{\mu+1}$. Then we define linear operator $L^{(2)}_1 = L^{(2)}_1[V^{(0)}_\mu, V^{(1)}_{\mu+1}]$ and let

$$H_{\mu+2} = ImL^{(2)}_1 \oplus N^{(2)}_{\mu+2}.$$

Then there is a polynomial $Y^2 = Y^{(2)}_1 + Y^{(2)}_2$, where $Y^{(2)}_1 \in KerL^{(1)}_1$ and $Y^{(2)}_2 \in H_2$ such that (10) is converted to

$$V^{(2)} = exp(adY^2)V^{(1)} = V^{(0)}_\mu + V^{(1)}_{\mu+1} + V^{(2)}_{\mu+2} + X^{(2)}_{\mu+3} + \cdots, \tag{11}$$

where $V^{(2)}_{\mu+2} \in N^{(2)}_{\mu+2}$. Then step by step, for each $m = 2, 3, \cdots, N$, we define a linear operator $L^{(m)}_1 = L^{(m)}_1[V^{(0)}_\mu, \cdots, V^{(m-1)}_{\mu+m-1}]$ and then find a polynomial $Y^m = Y^{(m)}_1 + \cdots + Y^{(m)}_m$, where $(Y^{(m)}_1, \cdots, Y^{(m)}_{m-1}) \in KerL^{(m-1)}_1$ and $Y^{(m)}_m \in H_m$ such that

$$\begin{aligned} V^{(m)} = \;& exp(adY^m)V^{(m-1)} \\ & = V^{(0)}_\mu + \cdots + V^{(m)}_{\mu+m} + X^{(m)}_{\mu+m+1} + \cdots, \end{aligned}$$

where $V^{(k)}_{\mu+k} \in N^{(k)}_{\mu+k}$ for $k = 1, \cdots, m$, and where $N^{(k)}_{\mu+k}$ is a complement to $ImL^{(k)}_1$ in $H_{\mu+k}$. Furthermore for $V^{(N)}$(denoted also as $V^{(N1)}$) and for each $k = 2, 3, \cdots$, we consider linear operator $L^{(N)}_k = L^{(N)}_k[V^{(0)}_\mu, \cdots, V^{(N-1)}_{\mu+N-1}]$, and find $Y^{N,k} = Y^{(N)}_k + \cdots + Y^{(N)}_{k+N-1}$ where $(Y^N_k, \cdots, Y^N_{k+N-2}) \in KerL^{(N-1)}_k$ and $Y^{(N)}_{k+N-1} \in H_{k+N-1}$ such that

$$\begin{aligned} V^{(Nk)} &= exp(adY^{N,k})V^{(N,(k-1))} \\ &= V^{(0)}_\mu + \cdots + V^{(N)}_{\mu+N} + V^{(N)}_{\mu+N+1} + \cdots + V^{(N)}_{\mu+k+N-1} + h.o.t. \end{aligned} \tag{12}$$

where $V^{(N)}_{\mu+N+j} \in N^{(N)}_{\mu+N+j}$ for each $j \geq 1$ and where $N^{(N)}_{\mu+N+j}$ is a complement to $ImL^{(N)}_{j+1}$ in $H_{\mu+N+j}$. Now the sequence of time one mappings defined by the sequence of polynomial vector fields $Y^1, \cdots, Y^N (= Y^{N,1}), Y^{N2}, \cdots$ change the given vector fields to an N-th order normal form. □

In what follows, we may always assume that all linear operators $L^{(m)}_k$ are defined by the same sequence of homogeneous polynomials $V_\mu, V_{\mu+1}, \cdots$.

Lemma 3.5

$$(0, Y_{k+1}, ..., Y_{k+m-1}) \in KerL^{(m)}_k \Longleftrightarrow (Y_{k+1}, ..., Y_{k+m-1}) \in KerL^{(m-1)}_{k+1}$$

Lemma 3.6

$$ImL_{k+1}^{(m)} \subset ImL_k^{(m+1)}, \quad \forall k, m \geq 1.$$

Proof. Note that

$$\begin{aligned} ImL_{k+1}^{(m)} \;=\; & \{X_{\mu+m+k} \mid \\ & \exists (Y_{k+1}, ..., Y_{k+m-1}) \in KerL_{k+1}^{(m-1)} \\ & \text{and } Y_{k+m} \in H_{k+m} \text{ such that} \\ & [Y_{k+1}, V_{\mu+m-1}] + ... + [Y_{k+m}, V_\mu] = X_{\mu+m+k}\}. \end{aligned}$$

Taking $Y_k = 0$. From Lemma 3.5, if such that

$$[Y_{k+1}, V_{\mu+m-1}] + ... + [Y_{k+m}, V_\mu] = X_{\mu+m+k},$$

then $(0, Y_{k+1}, ..., Y_{k+m-1}) \in KerL_k^{(m)}$ and

$$\begin{aligned} & [0, V_{\mu+m}] + [Y_{k+1}, V_{\mu+m-1}] + ... + [Y_{k+m}, V_\mu] \\ & = [Y_{k+1}, V_{\mu+m-1}] + ... + [Y_{k+m}, V_\mu] \\ & = X_{\mu+m+k}. \end{aligned}$$

Hence $X_{\mu+m+k} \in ImL_k^{(m+1)}$. □

Corollary 3.7

$$\dim N_{\mu+k+m}^{(m+1)} \leq \dim N_{\mu+k+m}^{(m)}, \quad \forall k, m \in \mathbb{N}.$$

Remark 3.8 *It is reasonable to set*

$$N_{\mu+k+m}^{(m+1)} \subset N_{\mu+k+m}^{(m)}, \quad \forall k, m \in \mathbb{N}.$$

Remark 3.9 *It is obvious that for a given formal vector field its N-th order normal form is simpler than its m-th order normal form if $m \leq N$.*

4 Unique normal forms

Definition 4.1
$$V = V_\mu + V_{\mu+1} + ... + V_{\mu+m} + ...$$
is called an infinite order normal form*, if* $V_{\mu+m} \in N^{(m)}_{\mu+m}$ *for* $\forall m \in \mathbb{N}$*, where* $N^{(m)}_{\mu+m}$ *is a complementary subspace to* $ImL^{(m)}_1$ *in* $H_{\mu+m}$ *and where* $L^{(m)}_1 = L^{(m)}_1[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ *for* $\forall m \in \mathbb{N}$*.*

Though in general we have infinitely many choices for the complementary space to the image of $L^{(m)}_1$ in $H_{\mu+m}$, in what follows, we assume that the choice of the complementary space $N^{(m)}_{\mu+m}$ to $ImL^{(m)}_1$ is fixed.

Theorem 4.2 *Let*

$$V = V_\mu + V_{\mu+1} + ... + V_{\mu+m} + ...$$

and
$$W = V_\mu + W_{\mu+1} + ... + W_{\mu+m} + ...$$
be both infinite order normal forms. If there exists a formal series $Y = Y_1 + Y_2 + ... + Y_m + ...$ *with* $Y_m \in H_m$ $(\forall m \in \mathbb{N})$ *such that* $(\Phi_Y)_* V = W$*, then*

$$V_{\mu+m} = W_{\mu+m} \quad \forall m \in \mathbb{N}.$$

Proof. Suppose it would not be the case. Then there exists an $m \in \mathbb{N}$ such that

$$V_{\mu+k} = W_{\mu+k} \ (1 \leq k \leq m-1), \quad V_{\mu+m} \neq W_{\mu+m}.$$

Recalling

$$\begin{aligned} W &= \exp(\mathrm{ad}Y)V \\ &= V + [Y, V] + \frac{1}{2!}[Y, [Y, V]] + ... + \frac{1}{n!}[Y, ..., [Y, V]...] + ..., \end{aligned}$$

we have

$$W_k = V_k + [Y, V]_k + \frac{1}{2!}[Y, [Y, V]]_k + ... + \frac{1}{n!}[Y, ..., [Y, V]...]_k + ...,$$

where $[Y, V]_k = [Y, V] \cap H_k$. Similarly for $[Y, ..., [Y, V]...]_k$. Notice that this infinite sum has in fact finitely many nontrivial terms, and hence the summation is well-defined. Therefore we have

$$[Y, V]_k + \frac{1}{2!}[Y, [Y, V]]_k + ... + \frac{1}{n!}[Y, ..., [Y, V]...]_k + ... = 0 \qquad (13)$$

for $\mu + 1 \leq k \leq \mu + m - 1$. It is easy to see that if $[Y, V] \neq 0$ and if the lowest degree of terms in $[Y, V]$ is l, then the lowest degree of terms in $[Y, ..., [Y, V]...]$ with n-fold bracket operations is $l + n - 1$. Hence from (13), we have

$$\begin{aligned} &[Y, V]_{\mu+1} = 0, \\ &[Y, V]_{\mu+2} + \frac{1}{2!}[Y, [Y, V]]_{\mu+2} = 0, \\ &\qquad\qquad \vdots \\ &[Y, V]_{\mu+m-1} + \frac{1}{2!}[Y, [Y, V]]_{\mu+m-1} + ... + \frac{1}{m!}[Y, ..., [Y, V]...]_{\mu+m-1} = 0. \end{aligned}$$

By induction, we have

$$[Y, V]_{\mu+1} = [Y, V]_{\mu+2} = ... = [Y, V]_{\mu+m-1} = 0,$$

and therefore

$$\begin{aligned} &[Y_2, V_\mu] + [Y_1, V_{\mu+1}] = 0, \\ &\qquad\qquad \vdots \\ &[Y_{m-1}, V_\mu] + [Y_{m-2}, V_{\mu+1}] + ... + [Y_1, V_{\mu+m-2}] = 0, \end{aligned}$$

namely, from Remark 3.2,

$$(Y_1, Y_2, ..., Y_{m-1}) \in Ker L_1^{(m-1)}.$$

Thus

$$W_{\mu+m} = V_{\mu+m} + [Y, V]_{\mu+m}.$$

Note that

$$[Y, V]_{\mu+m} = [Y_1, V_{\mu+m-1}] + ... + [Y_m, V_1],$$

This means $[Y, V]_{\mu+m} \in Im L_1^{(m)}$.

On the other hand, $V_{\mu+m}$ and $W_{\mu+m}$ are both in the same complementary space $N^{(m)}_{\mu+m}$ to $ImL^{(m)}_1$. Therefore

$$W_{\mu+m} - V_{\mu+m} = [Y, V]_{\mu+m} \in N^{(m)}_{\mu+m} \cap ImL^{(m)}_1 = \{0\},$$

and hence $W_{\mu+m} = V_{\mu+m}$. This contradiction shows that the conclusion of the theorem is true. $\square$

Corollary 4.3 *If*

$$V^{(N)} = V^{(0)}_{\mu} + V^{(1)}_{\mu+1} + ... + V^{(N)}_{\mu+N} + V^{(N+1)}_{\mu+N+1} + ...$$

and

$$W^{(N)} = V^{(0)}_{\mu} + W^{(1)}_{\mu+1} + ... + W^{(N)}_{\mu+N} + W^{(N+1)}_{\mu+N+1} + ...$$

are both N-th order normal form of (5), then

$$V^{(k)}_{\mu+k} = W^{(k)}_{\mu+k}, \qquad k = 1, ..., N.$$

5 A special case

In this section we assume that all linear operators $L^{(m)}_k$ are based one the same sequence of polynomials $V_\mu, V_{\mu+1}, \cdots$.

Proposition 5.1 *If there exists an $N \in \mathbb{N}$ such that*

holds, then

$$KerL^{(N+1)}_k = \underbrace{\{0\} \times ... \times \{0\}}_{m} \times KerL^{(N)}_{k+m}, \quad \forall k, m \in \mathbb{N} \tag{14}$$

and

$$ImL^{(N+m+1)}_k = ImL^{(N+1)}_{k+m}, \quad \forall k \in \mathbb{N}. \tag{15}$$

Proof. By assumption, (14) with $m = 1$ apparently holds. Suppose we have $X_{\mu+k+N+1} \in ImL_k^{(N+2)}$, namely, there exists $(Y_k, ..., Y_{k+N+1})$ satisfying

$$(Y_k, ..., Y_{k+N}) \in KerL_k^{(N+1)}, \ Y_{k+N+1} \in H_{k+N+1},$$
$$[Y_k, V_{\mu+N+1}] + ... + [Y_{k+N+1}, V_\mu] = X_{\mu+k+N+1}.$$

From assumption, $Y_k = 0$ and $(Y_{k+1}, ..., Y_{k+N}) \in KerL_{k+N}^{(N)}$. Hence

$$[Y_{k+1}, V_{\mu+N}] + ... + [Y_{k+N+1}, V_\mu] = X_{\mu+k+N+1}.$$

This implies that $X_{\mu+k+N+1} \in ImL_{k+1}^{(N+1)}$ and hence $ImL_k^{(N+2)} \subset ImL_{k+1}^{(N+1)}$.

Conversely, if we assume $X_{\mu+k+N+1} \in ImL_{k+1}^{(N+1)}$, namely, there exists $(Y_{k+1}, ..., Y_{k+N+1})$ satisfying

$$(Y_{k+1}, ..., Y_{k+N}) \in KerL_{k+1}^{(N)}, \ Y_{k+N+1} \in H_{k+N+1},$$
$$[Y_{k+1}, V_{\mu+N}] + ... + [Y_{k+N+1}, V_\mu] = X_{\mu+k+N+1}.$$

Then taking $Y_k = 0$, it holds that $(Y_k, ..., Y_{k+N+1}) \in KerL_k^{(N+1)}$ and apparently

$$[Y_k, V_{\mu+N+1}] + [Y_{k+1}, V_{\mu+N}] + ... + [Y_{k+N+1}, V_\mu] = X_{\mu+k+N+1}.$$

This implies $X_{\mu+k+N+1} \in ImL_k^{(N+1)}$, and hence $ImL_{k+1}^{(N+1)} \subset ImL_k^{(N+2)}$. Therefore

$$ImL_k^{(N+2)} = ImL_{k+1}^{(N+1)},$$

namely (15) holds for $m = 1$.

Suppose (14) and (15) hold for a fixed $m \geq 1$. Let $(Y_k, ..., Y_{k+N+m}) \in KerL_k^{(N+m+1)}$. Then

$$[Y_k, V_{\mu+N+m}] + ... + [Y_{k+N+m}, V_\mu] = 0. \tag{16}$$

Note that $(Y_k, ..., Y_{k+N+m-1}) \in KerL_k^{(N+m)}$. By induction hypothesis, $Y_k = 0, ..., Y_{k+m-1} = 0 \text{and} (Y_{k+m}, \cdots, Y_{k+m+N-1}) \in KerL_{k+m}^{(N)}$. Hence from (16),

$$L_{k+m}^{(N+1)}(Y_{k+m}, ..., Y_{k+N+m}) = 0,$$

or in other words,

$$(Y_{k+m}, ..., Y_{k+N+m}) \in KerL_{k+m}^{(N+1)}.$$

By assumption, $Y_{k+m}=0$ and $(Y_{k+m+1},...,Y_{k+m+N})\in KerL_{k+m+1}^{(N)}$. Hence

$$KerL_k^{(N+m+1)}\subset\underbrace{\{0\}\times...\times\{0\}}_{m+1}\times KerL_{k+m+1}^{(N)},\quad\forall k\in\mathbb{N}.$$

Conversely, take $Y_k=0,...,Y_{k+m}=0$ and $(Y_{k+m+1},...,Y_{k+m+N})\in Ker\ L_{k+m+1}^{(N)}$. Then

$$\begin{aligned}&L_k^{(N+m+1)}(Y_k,...,Y_{k+m+N})\\=&[Y_{k+m+1},V_{\mu+N-1}]+...+[Y_{k+m+N},V_\mu]\\=&0,\end{aligned}$$

namely,

$$\underbrace{\{0\}\times...\times\{0\}}_{m+1}\times KerL_{k+m+1}^{(N)}\subset KerL_k^{(N+m+1)}.$$

Therefore (14) holds for $m+1$ and for any $k\in\mathbb{N}$. In a similar way, (15) can be proved as in the case $m=1$. □

Corollary 5.2 *If there exists an N such that*

$$KerL_k^{(N+1)}=\{0\}\times KerL_{k+1}^{(N)},\quad\forall k\in\mathbb{N},$$

then an $(N+1)$-th order normal form must be an infinite order normal form.

Proof. From Proposition 5.1, we have

$$ImL_k^{(N+m+1)}=ImL_{k+m}^{(N+1)},\quad\forall k,m\in\mathbb{N}.$$

Hence we may set

$$N_{\mu+k+N+m}^{(N+m+1)}=N_{\mu+k+N+m}^{(N+1)}$$

as complementary subspaces to $ImL_k^{(N+m+1)}$ for $\forall k,m\in\mathbb{N}$. Thus, for any $m>N+1$,

$$N_{\mu+m}^{(m)}=N_{\mu+1+N+(m-N-1)}^{(N+(m-N-1)+1)}=$$

which implies that, if $V_{\mu+m}^{(N+1)}\in N_{\mu+m}^{(N+1)}$, then $V_{\mu+m}^{(N+1)}\in N_{\mu+m}^{(m)}$ for any $m\geq N+1$. The conclusion thus follows. □

Corollary 5.3 *If there exists an $N \in \mathbb{N}$ such that $ImL_k^{(N+m)} = ImL_{k+m}^{(N)}$ for any $k, m \in \mathbb{N}$, then the N-th order normal form is an infinite order normal form.*

Example 5.4 *If $KerL_k^{(1)} = \{0\}$, $\forall k \in \mathbb{N}$, then a first order normal form is also an infinite order normal form, and hence it is unique normal form of the original equation.*

6 The Bogdanov-Takens normal form: the case $\mu = 2, \nu = 1$

Baider and Sanders [BS2] gave unique normal forms for cases $\mu < 2\nu$ and $\mu > 2\nu$ of Bogdanov-Takens singularities. But the case $\mu = 2\nu$ is still unsolved. In this section we consider a special case, i.e., $\mu = 2, \nu = 1$. By using our method introduced above we give the unique normal form for this case.

We consider the following equation:

$$\begin{array}{rcl} \dot{x} & = & y + a_{11}xy + a_{02}y^2 + O(3), \\ \dot{y} & = & \alpha xy + \beta x^3 + b_{02}y^2 + O(3), \end{array} \tag{17}$$

where $\alpha, \beta \neq 0$.

Define $\delta : D_2 \to \mathbb{Z}$ by

$$\delta \begin{pmatrix} x^m y^n \\ 0 \end{pmatrix} = m + 2n - 1, \qquad \delta \begin{pmatrix} 0 \\ x^m y^n \end{pmatrix} = m + 2n - 2.$$

Then δ is a linear grading function with

$$\delta \begin{pmatrix} y \\ 0 \end{pmatrix} = \delta \begin{pmatrix} 0 \\ xy \end{pmatrix} = \delta \begin{pmatrix} 0 \\ x^3 \end{pmatrix} = 1, \text{ and } \delta \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = 1.$$

Let

$$V_1^{(0)} = \begin{pmatrix} y \\ \alpha xy + \beta y^3 \end{pmatrix}.$$

Then the equation (17) can be written as

$$V^{(0)} = V_1^{(0)} + V_2^{(0)} + \ldots + V_m^{(0)} + \ldots \tag{18}$$

where $V_m^{(0)} \in H_m^\delta$, $m = 1, 2, \cdots$.

Lemma 6.1 *The following vectors form a basis of the space H_m:*
For $m = 2k+1$,

$$\begin{pmatrix} 0 \\ x^{2k+3} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{2k+1}y \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ x^3y^k \end{pmatrix}, \begin{pmatrix} 0 \\ xy^{k+1} \end{pmatrix},$$

$$\begin{pmatrix} x^{2k+2} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{2k}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^2y^k \\ 0 \end{pmatrix}, \begin{pmatrix} y^{k+1} \\ 0 \end{pmatrix};$$

For $m = 2k+2$,

$$\begin{pmatrix} 0 \\ x^{2k+4} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{2k+2}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^2y^{k+1} \end{pmatrix}, \begin{pmatrix} 0 \\ y^{k+2} \end{pmatrix},$$

$$\begin{pmatrix} x^{2k+3}y \\ 0 \end{pmatrix}, \begin{pmatrix} x^{2k+1}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^3y^k \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{k+1} \\ 0 \end{pmatrix}.$$

In particular, $\dim H_m = m+3$.

Lemma 6.2

$$\begin{aligned}
\left[\begin{pmatrix} 0 \\ x^my^n \end{pmatrix}, V_1^{(0)}\right] &= \begin{pmatrix} -x^my^n \\ mx^{m-1}y^{n+1} + (n-1)\alpha x^{m+1}y^n + n\beta x^{m+3}y^{n-1} \end{pmatrix}, \\
\left[\begin{pmatrix} x^my^n \\ 0 \end{pmatrix}, V_1^{(0)}\right] &= \begin{pmatrix} mx^{m-1}y^{n+1} + n\alpha x^{m+1}y^n + n\beta x^{m+3}y^{n-1} \\ -\alpha x^my^{n+1} - 3\beta x^{m+2}y^n \end{pmatrix}.
\end{aligned}$$

Lemma 6.3 *Let*

$$Y_{2k+1} = \sum_{i=0}^{k+1} a_i \begin{pmatrix} 0 \\ x^{2k+3-2i}y^i \end{pmatrix} + \sum_{i=0}^{k+1} b_i \begin{pmatrix} x^{2k+2-2i}y^i \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
[Y_{2k+1}, V_1^{(0)}] &= \sum_{i=-1}^{k+1} \begin{pmatrix} 0 \\ x^{2k+3-2i}y^{i+1} \end{pmatrix} \{(2k+3-2i)a_i + ia_{i+1}\alpha \\
&\qquad +(i+2)a_{i+2}\beta - b_i\alpha - 3b_{i+1}\beta\} \\
&\quad + \sum_{i=-1}^{k} \begin{pmatrix} x^{2k+1-2i}y^{i+1} \\ 0 \end{pmatrix} \{(2k+2-2i)b_i + (i+1)b_{i+1}\alpha \\
&\qquad +(i+2)b_{i+2}\beta - a_{i+1}\},
\end{aligned}$$

where $a_i = 0,\ b_j = 0$ if $i, j < 0$ or $i, j > k+2$.

Lemma 6.4 *Let*

$$Y_{2k+2} = \sum_{i=0}^{k+2} a_i \begin{pmatrix} 0 \\ x^{2k+4-2i}y^i \end{pmatrix} + \sum_{i=0}^{k+1} b_i \begin{pmatrix} x^{2k+3-2i}y^i \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
[Y_{2k+2}, V_1^{(0)}] \;\; = \;\; & \sum_{i=-1}^{k+1} \begin{pmatrix} 0 \\ x^{2k+3-2i}y^{i+1} \end{pmatrix} \{(2k+4-2i)a_i + ia_{i+1}\alpha \\
& \quad +(i+2)a_{i+2}\beta - b_i\alpha - 3b_{i+1}\beta\} \\
& + \sum_{i=-1}^{k+1} \begin{pmatrix} x^{2k+1-2i}y^{i+1} \\ 0 \end{pmatrix} \{(2k+3-2i)b_i + (i+1)b_{i+1}\alpha \\
& \quad +(i+2)b_{i+2}\beta - a_{i+1}\},
\end{aligned}$$

where $a_i = 0,\ b_j = 0$ *if* $i, j < 0$ *or* $i, j > k+2$.

Using these results, we have a matrix representation for the adjoint operator $ad(V_1^{(0)})$,i.e.,$L_m^{(1)}$. Note that $L_m^{(1)} : H_m \to H_{m+1} : Y_m \mapsto [Y_m, V_1^{(0)}]$. Hence the matrix representation L of $L_m^{(1)}$ is given by an $(m+4) \times (m+3)$-matrix. Let

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}.$$

Then the submatrix L_3 is such that $-L_3$ is the identity matrix of the size $l = \left[\frac{m}{2}\right] + 2$, where $[q]$ stands for the integer part of q; The other three submatrices are (almost) tri-diagonal matrices. More specifically, they are given as follows:

For $m = 2k+1$,

$$L_1 = \begin{pmatrix}
-\alpha & \beta & 0 & & & \\
2k+3 & 0 & 2\beta & & & \mathbf{0} \\
& 2k+1 & \alpha & 3\beta & & \\
& & \ddots & \ddots & \ddots & \\
& & & \ddots & \ddots & (k+1)\beta \\
& & & & 3 & k\alpha \\
\mathbf{0} & & & & & 1
\end{pmatrix};$$

$$L_2 = \begin{pmatrix} -3\beta & & & & & \\ -\alpha & -3\beta & & & & \mathbf{0} \\ & -\alpha & -3\beta & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\alpha & -3\beta \\ \mathbf{0} & & & & & -\alpha \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ 2k+2 & \alpha & 2\beta & & & \mathbf{0} \\ & 2k & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & (k+1)\beta \\ \mathbf{0} & & & & 2 & (k+1)\alpha \end{pmatrix}.$$

For $m = 2k+2$,

$$L_1 = \begin{pmatrix} -\alpha & \beta & 0 & & & \\ 2k+4 & 0 & 2\beta & & & \mathbf{0} \\ & 2k+2 & \alpha & 3\beta & & \\ & & 2k & 2\alpha & \ddots & \\ & & & \ddots & \ddots & (k+2)\beta \\ \mathbf{0} & & & & 2 & (k+1)\alpha \end{pmatrix};$$

$$L_2 = \begin{pmatrix} -3\beta & & & & & \\ -\alpha & -3\beta & & & & \mathbf{0} \\ & -\alpha & -3\beta & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\alpha & -3\beta \\ \mathbf{0} & & & & & -\alpha \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ 2k+3 & \alpha & 2\beta & & & \mathbf{0} \\ & 2k+1 & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & (k+1)\beta \\ \mathbf{0} & & & & 1 & (k+1)\alpha \end{pmatrix}.$$

Remark 6.5 *In order to simplify the expression, we may assume that $\alpha = 1$ since we may make a suitable linear change of variables x, y, t in the equation (17) such that the coefficient of xy is changed to 1 and the coefficient of x^3 is changed to β/α^2 accordingly.*

Lemma 6.6 *For both $m = 2k+1$ and $m = 2k+2$, the first $k+3$ rows of matrix L can be reduced to the form*

$$\begin{pmatrix} 0 & \tilde{M} \end{pmatrix}$$

by a suitable row transformation. Here the matrix

$$\tilde{M} = (M_{ij}) \quad (-1 \le i \le k+1;\ 0 \le j \le k+1)$$

be given, using $\alpha = 1$, as follows:
[Case I] *For $m = 2k+1$,*

$$\begin{array}{rcrr} M_{i,i-1} & = & (2k+3-2i)(2k+4-2i) & (i = 1, \ldots, k+1) \\ M_{i,i} & = & (4k+5-4i)i - 1 & (i = 0, \ldots, k+1) \\ M_{i,i+1} & = & i(i+1) + \{(4i+6)k - (4i+3)i\}\beta & (i = -1, \ldots, k) \\ M_{i,i+2} & = & 2(i+1)(i+2)\beta & (i = -1, \ldots, k-1) \\ M_{i,i+3} & = & (i+2)(i+3)\beta^2 & (i = -1, \ldots, k-2) \end{array}$$

and the other entries are all zero.
[Case II] *For $m = 2k+2$,*

$$\begin{array}{rcrr} M_{i,i-1} & = & (2k+4-2i)(2k+5-2i) & (i = 1, \ldots, k+1) \\ M_{i,i} & = & (4k+7-4i)i - 1 & (i = 0, \ldots, k+1) \\ M_{i,i+1} & = & i(i+1) + \{(4i+2)k - (4i^2+3i-9)\}\beta & (i = -1, \ldots, k) \\ M_{i,i+2} & = & 2(i+1)(i+2)\beta & (i = -1, \ldots, k-1) \\ M_{i,i+3} & = & (i+2)(i+3)\beta^2 & (i = -1, \ldots, k-2) \end{array}$$

and the other entries are all zero.

For convenience, we denote $M_{i,i-1} = a_i$, $M_{i,i} = b_i$, $M_{i,i+1} = c_i + d_i\beta$, $M_{i,i+2} = e_i\beta, m_{i,i+3} = f_i\beta^2$ for both cases.

Lemma 6.7 *If β is not an algebraic number, then*

$$KerL_m^{(1)} = \{0\}, \qquad \forall m \in \mathbb{N}.$$

To show the lemma, it is sufficient to show that

$$\det M \neq 0$$

where $M = (M_{ij})_{0\leq i,j\leq k+1}$ is a submatrix of $\tilde{M}$.

First we consider the case II.

Lemma 6.8 *In the case II, we have*

$$\det M|_{\beta=0} \neq 0$$

Proof. Let D_l be the following subdeterminant:

$$D_l = \det\left(M_{ij}|_{\beta=0}\right)_{1\leq i,j\leq l}.$$

Then it is easy to see that

$$\det(M|_{\beta=0}) = (-1)\cdot D_{k+1}.$$

By induction we can show

$$D_l = \frac{(l+1)!(2k+1)!!}{(2k+1-2l)!!}.$$

In fact, it is true for $l = 1$ and 2. Since $M|_{\beta=0}$ takes a tri-diagonal form, we have

$$D_{l+1} = b_{l+1}\cdot D_l - c_l a_{l+1}\cdot D_{l-1}.$$

Therefore, using

$$\begin{aligned} b_{l+1} &= (4k+3-4l)(l+1)-1, \\ a_{l+1} &= (2k+2-2l)(2k+3-2l), \\ c_l &= l(l+1), \end{aligned}$$

we have

$$
\begin{aligned}
D_{l+1} &= \{(4k+3-4l)(l+1)-1\}\frac{(l+1)!(2k+1)!!}{(2k+1-2l)!!} \\
&\quad -l(l+1)(2k+2-2l)(2k+3-2l)\frac{l!(2k+1)!!}{(2k+3-2l)!!} \\
&= \frac{(l+1)!(2k+1)!!}{(2k+1-2l)!!}\{(4k+3-4l)l+(4k+2-4l)-l(2k+2-2l)\} \\
&= \frac{(l+1)!(2k+1)!!}{(2k+1-2l)!!}\cdot(l+2)(2k+1-2l) \\
&= \frac{\{(l+1)+1\}!(2k+1)!!}{\{2k+1-2(l+1)\}!!}.
\end{aligned}
$$

The conclusion is thus obtained.

It follows that

$$\det(M|_{\beta=0}) = -D_{k+1} = -(k+2)!(2k+1)!! \neq 0,$$

and hence the lemma is proved. □

Next we consider the case I. We introduce the same subdeterminant

$$D_l = \det\left(M_{ij}|_{\beta=0}\right)_{1\leq i,j\leq l}$$

for this case as well. Again, by induction, we can show

Lemma 6.9 *For the case I, we have*

$$D_l = \frac{(l+1)!2k!!}{(2k-2l)!!}.$$

From the lemma, we have

$$\det(M|_{\beta=0}) = -D_{k+1} = 0,$$

and hence we cannot conclude $\det M \neq 0$ immediately. We therefore differentiate $\det M$ with respect to β and will show that

$$\frac{\partial}{\partial\beta}\det M|_{\beta=0} \neq 0.$$

Let $M^{(l)}$ be the matrix given by differentiating the l-th column of the matrix M with respect to β. It thus takes the following form:

$$M^{(0)}|_{\beta=0} = \begin{pmatrix} 0 & 6k & 4 & 0 & \cdots & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & c_k \\ & \mathbf{0} & & & a_{k+1} & b_{k+1} \end{pmatrix};$$

$$M^{(l)}|_{\beta=0} = \begin{pmatrix} -1 & 0 & & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ a_1 & b_1 & c_1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \mathbf{0} & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & a_{l-1} & b_{l-1} & c_{l-1} & & & & \\ & & & & 0 & 0 & d_l & e_l & & \\ & & & & & a_{l+1} & b_{l+1} & c_{l+1} & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & & & & \ddots & \ddots & c_k \\ & & & & & & & & a_{k+1} & b_{k+1} \end{pmatrix};$$

$$M^{(k+1)}|_{\beta=0} = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & a_k & b_k & c_k \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

The determinants of these matrices are given as follows:

$$\begin{aligned} \det\left(M^{(0)}|_{\beta=0}\right) &= (-a_1)\left\{6k\delta_k - 4a_2\delta_{k-1}\right\}, \\ \det\left(M^{(l)}|_{\beta=0}\right) &= D_{l-1}(-a_{l+1})\left\{d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}\right\}, \quad 1 \leq l \leq k+1, \end{aligned}$$

where

$$\delta_m = \det\left(M_{ij}|_{\beta=0}\right)_{k+2-m \leq i,j \leq k+1}, \quad 1 \leq m \leq k-1,$$

and

$$\delta_0 = \delta_{-1} = \delta_{-2} = 1.$$

By induction, we can show:

Lemma 6.10

$$\delta_m = \frac{(2m-1)!!k!}{(k-m)!}, \quad 1 \leq m \leq k-1$$

From these formulas, we shall compute

$$\begin{aligned} \frac{\partial}{\partial \beta} \det M_{\beta=0} \quad &= \quad (-a_1)\left\{6k\delta_k - 4a_2\delta_{k-1}\right\} \\ &\quad + \sum_{l=1}^{k-1}(-a_{l+1}D_{l-1})\left\{d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}\right\} \\ &\quad +(-a_{k+1}D_{k-1})d_k. \end{aligned}$$

First we compute the second term. Since

$$\begin{aligned} D_{l-1} &= \frac{l!(2k)!!}{(2k+2-2l)!!} = \frac{l!2^k k!}{2^{k+1-l}(k+1-l)!} = \frac{2^{l-1}k!l!}{(k+1-l)!}, \\ \delta_{k-l} &= \frac{(2k-1-2l)!!k!}{l!} = \frac{(2k-1-2l)!k!}{l!2^{k-1-l}(k-1-l)!} = \frac{(2k-2l)!k!}{2^{k-l}l!(k-l)!}, \\ \delta_{k-l-1} &= \frac{(2k-3-2l)!!k!}{(l+1)!} = \frac{(2k-3-2l)!k!}{(l+1)!2^{k-2-l}(k-2-l)!} \\ &= \frac{(2k-2-2l)!k!}{2^{k-1-l}(l+1)!(k-1-l)!}, \end{aligned}$$

we have

$$d_l D_{l-1}\delta_{k-l} - a_{l+2}e_l D_{l-1}\delta_{k-l-1} = -(2k-5l)\frac{2^{2l-1}(k!)^2(2k-2l)!}{(k+1-l)!(k-l)!2^k}.$$

Hence

$$-a_{l+1}D_{l-1}\left(d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}\right) = (2k-5l)\frac{2^{2l-1}(k!)^2(2k+2-2l)!}{(k+1-l)!(k-l)!2^k}.$$

We need to compute

$$\sum_{l=1}^{k-1}(2k-5l)\frac{2^{2l-1}(2k+2-2l)!}{(k+1-l)!(k-l)!}\cdot\frac{(k!)^2}{2^k}.$$

Let $i = k+1-l$ and

$$A_p(i) = \frac{(2i)!}{i!i!}i^p\cdot 4^{k-i}, \quad p=1,2,3,$$

for simplicity of notation. Then $l = k+1-i$ and

$$(2k-5l)\frac{2^{2l-1}(2k+2-2l)!}{(k+1-l)!(k-l)!}\cdot\frac{(k!)^2}{2^k} = -\frac{(3k+5)(k!)^2}{2^{k-1}}A_1(i)+\frac{5(k!)^2}{2^{k-1}}A_2(i).$$

Therefore we need to compute

$$\sum_{i=2}^{k}A_p(i) \quad \text{for} \quad p=1,2.$$

Lemma 6.11 *(1)* $A_1(i+1)-A_1(i)=\frac{1}{2}A_0(i)$

(2) $A_2(i+1)-A_2(i)=\frac{3}{2}A_1(i)+\frac{1}{2}A_0(i)$

(3) $A_3(i+1)-A_3(i)=\frac{5}{2}A_2(i)+2A_1(i)+\frac{1}{2}A_0(i)$

Proof. A simple computation shows

$$A_p(i+1)-A_p(i)=\frac{(2i)!}{i!i!}4^{k-i}\cdot\left\{\left(i+\frac{1}{2}\right)(i+1)^{p-1}-i^p\right\}.$$

For $p=1,2,3$, we get

$$\begin{aligned}
\left(i+\frac{1}{2}\right)(i+1)^{1-1}-i^1 &= \frac{1}{2}i^0,\\
\left(i+\frac{1}{2}\right)(i+1)^{2-1}-i^2 &= \frac{3}{2}i^1+\frac{1}{2}i^0,\\
\left(i+\frac{1}{2}\right)(i+1)^{3-1}-i^2 &= \frac{5}{2}i^2+2i^1+\frac{1}{2}i^0,
\end{aligned}$$

and hence the lemma follows. □

Summing them up from $i = 2$ to k, we have

$$
\begin{aligned}
\sum_{i=2}^{k} A_1(i) &= \frac{2}{3}\{A_2(k+1) - A_2(2) - A_1(k+1) + A_1(2)\} \\
&= \frac{2}{3}\left\{\frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{k(k+1)}{4} - 3 \cdot 4^{k-1}\right\}; \\
\sum_{i=2}^{k} A_2(i) &= \frac{2}{5}\left\{A_3(k+1) - \frac{4}{3}A_2(k+1) + \frac{1}{3}A_1(k+1) - A_3(2)\right\} \\
&\quad + \frac{2}{5}\left\{\frac{4}{3}A_2(2) - \frac{1}{3}A_1(2)\right\} \\
&= \frac{2}{5}\left\{\frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{k(k+1)(k+\frac{2}{3})}{4} - 5 \cdot 4^{k-1}\right\}.
\end{aligned}
$$

From these expressions, we have

$$
-\frac{(3k+5)(k!)^2}{2^{k-1}} \sum_{i=2}^{k} A_1(i) + \frac{5(k!)^2}{2^{k-1}} \sum_{i=2}^{k} A_2(i) = \frac{(2k+1)!}{2^{k-1}} \cdot (-k) + 2^k \cdot (k!)^2 \cdot 3k.
$$

Therefore we finally obtain

$$
\frac{\partial}{\partial \beta} \det M_{\beta=0} = \frac{(2k+1)!k(2k+3)}{2^{k-1}} > 0,
$$

and hence we conclude $\det M \neq 0$.

Theorem 6.12 *If β/α^2 is not an algebraic number, then the first order normal form of Eqn.(17) with respective to the grading function δ is unique, which is*

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \alpha x y + \beta x^3 + \textstyle\sum_{m=4}^{\infty} a_m x^m.
\end{aligned} \tag{19}
$$

Proof. The uniqueness of the first order normal form follows from Corollary 5.2, Example 5.4 and Lemma 6.7. A simple calculation shows that $span x^{m+3}\partial y$ is a complement to $Im L_m^{(1)}$ for each $m \in \mathbb{N}$. Then the first order normal form (19) is obtained.

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