Chaotic solutions in slowly varying perturbations of Hamiltonian systems with applications to shallow water sloshing*

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Abstract

We study a slowly varying planar Hamiltonian system modeling shallow water sloshing. Using the Conley index theory for fast-slow systems of ODEs, we prove the existence of complicated dynamics in the system which is described in terms of symbolic sequences of integers. This includes the solutions proven by Hastings and McLeod as well as those conjectured by them.

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1 Introduction

Slowly varying Hamiltonian systems model various kinds of physical phenomena, which sometimes exhibit very complicated behaviors (e.g. [15, 3, 4, 2]). The purpose of this paper is to show that such systems can have rich variety of solutions whose behaviors are described in terms of a certain type of symbolic coding.

The equation that we consider is a slowly varying planar Hamiltonian system with a higher order perturbation given as follows:

$$\frac{du}{dt} = J\nabla H(u, \varepsilon t) + \varepsilon^2 h(u, \varepsilon t, \varepsilon), \quad u \in \mathbb{R}^2, \tag{1.1}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic structure on the plane and $\varepsilon > 0$ is taken to be small enough. The Hamiltonian function H and the perturbation term h are assumed to be sufficiently smooth (HOW SMOOTH?) One can rewrite this system into the form of a multi-time scale vector field on $\mathbb{R}^3$ in the following way:

$$\dot{u} = J\nabla H(u, \lambda) + \varepsilon^2 h(u, \lambda, \varepsilon), \tag{1.2}$$

$$\dot{\lambda} = \varepsilon. \tag{1.3}$$

For the study of chaotic dynamics for time-dependent Hamiltonian systems, one often assumes that the Hamiltonian is time periodic, thereby reducing it to the study of the Poincaré map associated to the time period. This approach is not possible here since the Hamiltonian is not necessarily time periodic but has merely a kind of oscillatory character, which will be made precise later.

We make the following two hypotheses for the system (1.1):

HYPOTHESIS

(H1) For $\varepsilon = 0$, the system (1.2) reduces to a one-parameter family of 2-dimensional Hamiltonian systems

$$\frac{du}{dt} = J\nabla H(u, \lambda). \tag{1.4}$$

We suppose that, for each λ, the Hamiltonian system (1.4) has two equilibrium points $A(\lambda)$ and $B(\lambda)$ which depend C^1 in λ, and that $B(\lambda)$ is hyperbolic with a homoclinic orbit that enclose, in u-plane, the other equilibrium point $A(\lambda)$.

We denote $\mathcal{A} = \{A(\lambda)\}$ and $\mathcal{B} = \{B(\lambda)\}$, respectively. Notice that the curve $\mathcal{B}$ is a normally hyperbolic invariant manifold whose stable and unstable manifolds close up and form a surface of homoclinic orbits.

(H2) Let $S(\lambda)$ denote the area surrounded by the homoclinic orbit in the u-plane for each λ. We assume that the function $S(\lambda)$ is C^1 and its derivative $S'(\lambda)$ is infinitely oscillating, in the sense that there exists a positive constant s_0 and a sequence of disjoint closed intervals $\Lambda_i = [\lambda_i^-, \lambda_i^+]$ such that $\lambda_i^+ < \lambda_{i+1}^-$, $S'(\lambda_{2i}^-) = s_0$, $S'(\lambda_{2i}^+) = -s_0$ and $S'(\lambda_{2i+1}^-) = -s_0$, $S'(\lambda_{2i+1}^+) = s_0$ for all $i \in \mathbb{Z}$.

Notice that if $H(u, \lambda)$ is periodic in λ and if $S(\lambda)$ is not a constant function, then $S'(\lambda)$ is always infinitely oscillating.

Definition 1.1 A solution $u(t), \lambda(t))$ of the equations (1.2, 1.3) is said to *oscillate k times* over an interval $\Lambda = [\lambda^-, \lambda^+]$, if the winding number of the solution with respect to the curve $\mathcal{A}$ over Λ is equal to k. Here the winding number is defined as the homotopy class of the following closed loop l in the fundamental group of $\mathbb{R}^3 \setminus \mathcal{A}$ which is isomorphic to $\mathbb{Z}$:

$$l = \left(\cup_{\lambda(t)\in\Lambda}\{(u(t), \lambda(t))\} \cup B(\lambda)\right) \bigcup \left(\cup_{\lambda(t)\in\partial\Lambda}\overline{u(t)\ B(\lambda(t))}\right).$$

Now we are ready to state the main theorem of this paper.

Main Theorem *Assume the hypotheses (H1) and (H2) are satisfied, and hence there exists a sequence of disjoint closed intervals $\{\Lambda_i\}$ as in (H2). Suppose the length of Λ_i is bounded from below by a positive constant. Then, for a given integer $K > 0$, there exists $\bar{\epsilon} > 0$ such that for any $0 < \epsilon < \bar{\epsilon}$ and for any sequence of integers $\{\sigma_i\}_{i\in\mathbb{Z}}$ with the property that*

$$\sigma_{2i} \in \{0, \ldots, K\}, \quad \textit{and} \quad \sigma_{2i+1} \in \{0, 1\},$$

there exists a solution of (1.2)-(1.3) which oscillates σ_i times over Λ_i.

This theorem asserts that under these hypotheses, there exists a set of solutions which have certain number of oscillations over the given intervals Λ_i which are prescribed in terms of symbolic sequences of integers, and therefore one may say that the system has some kind of complicated or "chaotic" dynamics. Even if (H2) fails and the function $S'(\lambda)$ oscillates only finitely many times, one can still have similar conclusion if one replaces the infinite sequence of integers by a finite sequence of integers.

This work is motivated by a result of Hastings and McLeod [3], in which they studied some complicated dynamics in a problem of shallow water sloshing. We will discuss this example in Section 2. The proof of the main theorem is given in Section 3. The proof uses the Conley index theory and in particular, recently developed method adapted for singularly perturbed ODEs, which are summarized in Appendix A. Section 4 is devoted for discussions on related results and concluding remarks.

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2 Example: shallow water sloshing

H. Ockendon, J. R. Ockendon and A. D. Johnson studied the problem of shallow water sloshing [10]. This is a two-dimensional irrotational fluid motion in a rectangular tank of inviscid fluid with horizontal oscillatory forcing. Assuming that forcing is time periodic, they obtained a partial differential equation model for this problem. They also derived a fourth order ordinary differential equation, as a first order approximation of the PDE model, which is as follows:

$$\frac{1}{3}\kappa^2 f''''(t) - (\rho - \frac{1}{3}\kappa^2)f''(t) - 3f'(t)f''(t) = \frac{2}{\pi}\sin t, \tag{2.1}$$

where the leading term of the velocity potential is given as $f(t-x)+f(t+x)$, if the amplitude of the oscillation is very small with respect to the horizontal length of the rectangular tank, and the constants κ and ρ are determined from the size of the tank as well as the amplitude of the forcing. In particular, we are interested in the case the constant κ is small. Note that in the papers of [10] and [3] the constant λ is used in place of ρ above. We used different notation in order to avoid confusion as λ is used as the slow variable.

S. P. Hastings and J. B. McLeod ([3]) studied this ODE model when the constant $\rho = \frac{1}{3}\kappa^2$. In this case one can integrate the equation once and, by putting $g = f'$, reduce the ODE to the following second order equation:

$$\frac{1}{3}\kappa^2 g''(t) = \frac{3}{2}g(t)^2 - \frac{2}{\pi}\cos t + C,$$

where C is an integration constant, or equivalently, it can be written as a fast-slow system of the form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 - 1 - c - \cos(\epsilon t) \end{aligned}$$

after appropriate rescaling of variables. Notice that this is a slowly varying time-periodic planar Hamiltonian system, and if $c > 0$, it satisfies the hypothesis (H1) given in Section 1.

Using what they call "simple shooting method", Hastings and McLeod noticed that any solution $x(t)$ of this equation cannot have a minimum value in the range $-\sqrt{c} < x < \sqrt{c}$, and hence relative minima of the solution are sharply distinguished as positive (larger than $\sqrt{c}$) or negative (less than $-\sqrt{c}$). This allows them to introduce a symbolic coding of solutions in terms of the number of successive positive minima and negative minima. Using this idea, they have proved, among other things, the existence of solutions which are coded by arbitrary sequence of symbols 1's (negative minima) and 0's (positive minima). Moreover these solutions have minima corresponding to the symbols in each 2π-interval of $\lambda = \epsilon t$. There also proven to exist "subharmonic" solutions which have more negative minima with one or no positive minima in each 2π-interval of λ.

Numerically, Hastings and McLeod seemed to have observed the existence of more complicated solutions whose existence is not proven in [3]. In fact, they conclude the paper by the following concluding remark:

> .... It seems likely that these are infinite families of subharmonics with n spikes near even multiples of π and zero or one spike near odd multiples of π, ...

Using the main theorem of this paper, one can in fact prove the existence of such families of solutions. In order to apply the theorem to the system, one has to verify another hypothesis (H2). From a simple calculation, the area funstion of the planar Hamiltonian system is given by

$$S(\lambda) = \frac{24\sqrt{2}}{5}(1 + c + \cos\lambda)^{3/4},$$

and hence it is not a constant function and its derivative $S'(\lambda)$ is infinitely oscillating, as it is a periodic function of λ. Observe that the zeroes of $S'(\lambda)$ are multiples of π, and hence one can take the intervals Λ_{2i} as a compact neighborhood of an even multiple of π, whereas Λ_{2i+1} a compact neighborhood of an odd multiple of π. Therefore, from Main Theorem, we conclude that, given an arbitrary sequence of integers $\{\sigma_i\}$ with the property that $\sigma_{2i} = 0, 1, ..., K$ for some K and $\sigma_{2i+1} = 0, 1$, there exists a solution which oscillates σ_{2i} times in a neighborhood of an even multiple of π and σ_{2i+1} times in a neighborhood of an odd multiple of π in a prescribed manner. This assertion is exactly what is conjectured at the end of [3].

In fact, the same analysis can be applied to the original equation (2.1) derived by [10]. Integrating (2.1) once and rewriting it as in the form of the system

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= \frac{3}{2}y^2 + (\rho - \epsilon^2)y + c - \frac{2}{\pi}\cos(\epsilon t) \end{aligned}$$

where $y(t) = f'(\epsilon t)$ and ϵ is a small constant proportional to κ, one again has a family of slowly varying planar Hamiltonian systems which satisfies the assumptions (H1) and (H2). The Hamiltonian function for this system is given by

$$H(y, z, \lambda) = \frac{1}{2}z^2 - \frac{1}{2}y^3 - \frac{1}{2}(\rho - \epsilon^2)y^2 - cy + \frac{2}{\pi}y\cos\lambda$$

for each fixed λ. Straightforward calculation shows that the area function $S(\lambda)$ is given by

$$S(\lambda) = \frac{8}{15}\left\{(\rho\epsilon^2)^2 - 6\left(c - \frac{2}{\pi}\cos\lambda\right)\right\}^{5/4},$$

and hence the derivative $S'(\lambda)$ has a zero at λ being an integer multiple of π, and is decreasing if λ is an odd multiple of π, whereas increasing if λ is an even multiple of π. Therefore we basically have the same conclusion about the existence of solutions that behave according to symbolic sequences of $\{\sigma_i\}$ as before.

Furthermore, it is not necessary to assume that the forcing term is exactly periodic in time. Therefore even if the original forcing term is not exactly periodic but is merely oscillating in such a way that it satisfies the condition (H2), one can still apply the Main Theorem to this problem. This may be a more realistic formulation of the original problem of sloshing.

Hasting and McLeod studied similar problems in [4] and [2], to which the main theorem of this paper is equally applicable. We will discuss more about the difference between our approach and those taken by Hasting et al., or by Wiggins in Section 4.

3 Proof of Main Theorem

3.1 Splitting of homoclinic surfaces

Consider

$$\begin{aligned} \dot{u} &= J\nabla H(u,\lambda) + \varepsilon^2 h(u,\lambda,\varepsilon) \\ \dot{\lambda} &= \varepsilon \end{aligned}$$

As assumed in (H1), this system has a surface of homoclinic orbits when $\varepsilon = 0$. Since the curve B consisting of equilibrium points $B(\lambda)$ for all $\lambda \in \mathbb{R}$ is normally hyperbolic, the stable and unstable manifolds $W^s(B)$ and $W^u(B)$ persist for non-zero but small ε. One can then see how these maniflds intersect by measuring the splitting distance as follows. Take a transverse section to the surface of homoclinic orbits for $\varepsilon = 0$. This remains to be a section to $W^s(B)$ and $W^u(B)$ if $\varepsilon > 0$ is small enough. Let $d(\lambda, \varepsilon)$ be the signed distance measured from $W^u(B)$ to $W^s(B)$ in the intersection of the cross section and the plane given by each λ.

Theorem 3.1 ([12, 15, 14, 11]) *The signed distance $d(\lambda,\varepsilon)$ of $W^u(B)$ and $W^s(B)$ defined above satisfies the following asymptotic expression:*

$$d(\lambda,\varepsilon) \approx \varepsilon(S'(\lambda) + O(\varepsilon)).$$

In particular, if $S'(\lambda)$ takes non-zero values with different sign at λ^+ and λ^-, then, for sufficiently small $\varepsilon > 0$, $W^u(B)$ and $W^s(B)$ intersect topologically transversely over the interval between λ^- and λ^+.

Note that the formula

$$d(\lambda,\epsilon) \approx \epsilon(S'(\lambda) + O(\epsilon))$$

is true but the constants depend on the choice of cross sections where the distance is measured. In particular, the term $O(\epsilon)$ is proportional to some constant which depends on the location of the section, and therefore, if the cross section is taken too close to the hyperbolic saddle B, the constant may be very large so that the second term dominates the first term for small but non-zero ϵ. Even in such a case, the formula remains true if one chooses smaller value of ϵ.

This is a version of so-called Melnikov-type theorems. The idea of Melnikov integrals was for the first time applied to slowly varying ODEs by Robinson [12], and then further developed in [15], [14], [11], and others. This particular formulation as well as its proof can be found in [11].

3.2 Existence of infinitely many connecting orbits

Suppose there exist $\lambda^- < \lambda^+$ such that the function $S'(\lambda)$ takes non-zero values with different sign. Take an open interval Λ in the λ-axis containing $[\lambda^-, \lambda^+]$ and let $g(\lambda)$ be a smooth function which vanishes at $\lambda^\pm$ with $g'(\lambda^-) > 0$ and $g'(\lambda^+) < 0$ and which does not vanish elsewhere in Λ.

Consider a modified equation as follows:

$$\begin{aligned} \dot{u} &= J\nabla H + \varepsilon^2 h - \delta S'(\lambda)\nabla H, \\ \dot{\lambda} &= \varepsilon g(\lambda). \end{aligned}$$

Note that, from the above conditions of g, the points on B with $\lambda = \lambda^-, \lambda^+$ are hyperbolic equilibrium points. $\delta S'(\lambda)\nabla H$ is an artificial dissipation term; this is introduced in order to make the splitting distance of $W^u(B)$ and $W^s(B)$ even bigger.

We are interested in finding orbits connecting the equilibria $B^\pm$ in B at $\lambda = \lambda^\pm$. Since we only look for topologically distinguishable connecting orbits, we will measure the winding number with respect to the curve A, defined in §1. Below we only consider the case $S'(\lambda^-) > 0$ and $S'(\lambda^+) < 0$i for simplicity, as the other case is given by reversing the time and the λ-axes if we are allowed to make $\varepsilon < 0$. Instead we need to take into account of positive and negative ε.

Theorem 3.2 ([6]) *(1) For sufficiently small $\varepsilon > 0$, there exist at least 2 but finitely many connecting orbits with distinct winding numbers;*

(2) For sufficiently small $\varepsilon < 0$, there exist infinitely many connecting orbits with distinct winding numbers.

Proof of this theorem goes as follows: First we take an isolating neighborhood which is homeomorphic to a 3-disk and which contains the whole surface of homoclinic orbits over the interval Λ, and remove a tubular neighborhood of the curve A. It then becomes homeomorphic to a fattened cylinder, or the product of an annulus and an interval, whose universal covering space is homeomorphic to a 3-disk. In this covering space, there are infinitely many lifts of the piece of the curve B over the interval Λ. Let B_i be those lifts. For each lift B_i, there are also the corresponding lift of the equilibria $B_i^\pm$. From the construction, the flow lifted on the covering space is monotone in such a way that an orbit passing near some B_i may pass near B_j with $j > i$, but never do so with $j < i$. This comes from the assumption that the orbits in the base space rotates in one direction with respect to the curve A. Therefore we look for connecting orbits from B_i^- to B_j^+ with $j \geq i$ for $\varepsilon > 0$, and those from B_i^+ to B_j^- for $\varepsilon < 0$.

The difference between the two assertions of the theorem for positive and negative ε is completely topological. Intuitively, readers can easily be convinced by the figures in [6]. Noticing this, in order to find those connecting orbits, we use the Conley index theory, and in particular the theory of transition matrices. The transition matrix is a matrix whose non-zero (i, j)-entry shows the existence of the connecting orbit from $B_i^\pm$ to $B_j^\mp$. The essential part of the proof is very simple: one computes the transition matrix of the covering space for $\varepsilon < 0$, using the information of the flow on the base space. It turns out that its non-zero (i, j) entries are such that the positive integers $j - i$ are only finitely many. Notice that the integer $j - i$ is nothing but the winding number of the connecting orbit once it is projected down to the base space. On the other hand, it is known that the transition matrix for $\varepsilon > 0$ is given by taking the inverse of that for $\varepsilon < 0$. This implies that in this case one must have infinitely many integers $j - i$ from non-zero entries of the transition matrix for $\varepsilon > 0$. For the detail of the argument, see [6].

One of the important consequences of the proof is that each connecting orbit found by the above argument must have its isolating neighborhood, since it is detected by the Conley index theory which is based on isolated invariant sets and their isolating neighborhoods. Moreover, the transition matrix is viewed as a linear map acting on the sum of the homology group of the individual Morse components (the equilibria $B_i^{\pm}$ in our case) to itself, and therefore if a connecting orbit from $B_i^{\pm}$ to $B_j^{\mp}$ exists, the transition matrix carries the generator of the homology Conley index of $B_i^{\pm}$ to that of $B_j^{\mp}$ non-trivially. It is this information we will make use of later in order to construct the symbolic dynamics. Notice that since this is a purely topological information, it does not depend on the choice of the funtion g. One can hence choose g so that it is identically equal to 1 on an open interval slightly smaller than $[\lambda^-, \lambda^+]$.

3.3 Concatenation of connecting orbits

Let us go back to our original problem (1.2)-(1.3). Recall from the assumption (H2) that there are intervals $\Lambda_i = [\lambda_i^-, \lambda_i^+]$ such that $S'(\lambda_i^-) > 0$ and $S'(\lambda_i^+) < 0$. Those intervals are disjoint. Main partof the proof is to show that, given a finite sequence $\sigma^J = \{\sigma_{-J}, \ldots, \sigma_J\}$ of admissible symbols, namely, the symbols satisfying $\sigma_{2i-1} \in \{0, 1\}$, $\sigma_{2i} \in \{0, \ldots, K\}$, there exists an orbit which behaves over the intervals Λ_i with $i = -J, ..., J$ according to the finite symbol sequence.

Consider the equation

$$
\begin{aligned}
\dot{u} &= J\nabla H + \varepsilon^2 h - \delta S'(\lambda)\nabla H, \\
\dot{\lambda} &= \varepsilon \tilde{g}_J(\lambda),
\end{aligned}
$$

where $\tilde{g}_J$ vanishes at λ_{-J}^- and λ_J^+ with $g'(\lambda_{-J}^-) > 0$ and $g'(\lambda_J^+) < 0$, and $g = 1$ everywhere in a slightly small open interval of $[\lambda_{-J}^-, \lambda_J^+]$. Similarly to the previous subsection, we can lift this equation onto an appropriate covering space.

If one furthermore modify the equation as in the previous subsection so that the points on the curve B corresponding to the boundary points of each Λ_i, one can find a connecting orbit which oscillates σ_i times over Λ_i. Moreover, the connecting orbit carries a homology generator of the fast dynamics at λ_i^- to that at λ_i^+ in a non-trivial manner, and this correspondence of the generators is given by a non-zero entry of the transition matrix. Since the generators of the fast dynamics can be defined without creating virtual equilibria on B at $\lambda_i^{\pm}$, the homology information about the orbits that oscillate σ_i times over the interval Λ_i is retained.

This argument shows that for each Λ_i one can construct a σ_i-fold covering space in which the transition matrix has a non-zero entry corresponding to orbits oscillating Λ_i times over Λ_i. This part of the covering space satisfies the definition of the box defined in Appendix as well as a sufficiently small tubular neighborhood of a lift of the curve B which connects an exit part of the box over Λ_i and the entrance side of the next box Λ_{i+1} satisfies the tube also defined in Appendix. Therefore we have a collection of boxes and tubes which are compatible in the sense of Definition A.4. If we take an isolating neighborhood of the (artificially created) hyperbolic equilibrium point on B at λ_{-J}^- and at λ_J^+, respectively, they satisfy the condition of caps, one being a repelling cap and another an attracting cap, and hence we have constructed a tube-box-cap collection.

We apply the TBC collection theorem in Appendix and conclude the existence of a connecting orbit $(u_J(t), \lambda_J(t))$ that oscillates according to the finite symbolic sequence σ^J.

Theorem 3.3 ([1]) *(1) The union N_J consisting of the tubes, boxes and caps is an isolating neighborhood for sufficiently small $\varepsilon > 0$.*

(2) $(\mathrm{Inv}C(R), \mathrm{Inv}C(A))$ *is an attractor-repeller pair for* $\mathrm{Inv}N_J$.

(3) If the composition of the transition maps $T_J \circ \cdots \circ T_{-J} \neq 0$, then there exists a connecting orbit $(u_J(t), \lambda_J(t))$ from $\mathrm{Inv}C(R)$ *to* $\mathrm{Inv}C(A)$.

In our case, all T_i are the non-zero entries of the transition matrix on each Λ_i, and therefore we have proven that there exists a connecting orbit.

It remains to remove the artificial dissipation term given by δ.

Key Lemma *Let N_J be an isolating neighborhood for the connecting orbit $(u_J(t), \lambda_J(t))$ for some $\bar{\delta} > 0$. Then it is so for any $\delta \in [0, \bar{\delta}]$.*

Proof. Suppose N_J fails to be an isolating neighborhood for some $\delta > 0$. Since the only invariant set inside N_J is either the hyperbolic equilibrium in the caps or possibly a connecting orbit between them, this supposition means that there exists a connecting orbit $(u_\delta(t), \lambda_\delta(t))$ which has an inner tangency at a point of the boundary of N_J.

From [13], the flow inside a tube is C^1-linearizable. Since the size of the tube is of $O(1)$, orbits staying entirely in a tube and its adjacent box are of $O(e^{-1/\varepsilon})$-close to the unstable manifold of the normally hyperbolic curve B when they leave the tube. In fact, under the C^1-linearizing coordinates, the flow in the tube can simply be written as

$$\dot{u} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u, \quad \dot{\lambda} = \varepsilon.$$

One can also obtain the same conclusion at the first hitting point of the same orbit at the cross section of the unperturbed homoclinic surface on which the splitting distance of $W^u(B)$ and $W^s(B)$ is measured. This claim follows from the fact that it takes only of $O(1)$ amount of time (namely the $O(\varepsilon)$ length along the λ-axis) from the end of the tube to the first hitting point of the cross section. The standard Gronwall type estimate then proves the assertion.

From the construction of the neighborhood N_J, after a finitely many turns around the curve A, the connecting orbit must come back to a tubular neighborhood of the curve B and stays there for $O(1)$ amount of time. Applying the same argument to the time reversed flow, the orbit is $O(e^{-1/\varepsilon})$-close to $W^s(B)$ at the landing time, and hence at the hitting point of the cross section. Therefore the distance between $W^u(B)$ and $W^s(B)$ in the cross section must be of $O(e^{-1/\varepsilon})$. However, we know that the splitting distance is of $O(\varepsilon)$, which is a contradiction. Therefore we conclude that no orbit in N_J has a point of inner tangency at the boundary, and hence it remains to be isolated. $\square$

Remark 3.4 *The isolating neighborhood was chosen in such a way that the homoclinic surface at $\varepsilon = 0, \delta = 0$ is $O(1)$ away from its top and bottom boundaries. Therefore the only way to lose isolation is at the side boundary.*

The Key Lemma enables us to compute the Conley homology index of N_J with $\delta = 0$ and $\varepsilon > 0$, which implies the existence of a connecting orbit in N_J even for $\delta = 0$.

Let $\sigma = \{\sigma_i\}_{i\in\mathbb{Z}}$ be an admissible infinite sequence, and let $\sigma^J = \{\sigma_{-J}, \ldots, \sigma_J\}$ be its finite truncation from σ_{-J} to σ_J.

3.4 Final step

Let $(u_J(t), \lambda_J(t))$ be a connecting orbit which behaves accordingly to σ^J. Choosing a subsequence, if necessary, one obtains a convergent sequence $(u_J(t), \lambda_J(t))$ and its limit $(u(t), \lambda(t))$ is a desired solution.

This completes the proof of Main Theorem.

4 Discussion

Our main result shows the existence of conjectured symbolic sequences of Hastings and McLeod [3]. Also it can be applied to other examples discussed in [4, 2].

Hastings and McLeod approach is based on a shooting method using information of solutions. Their method is elementary, but requires good guess for the expected structure of solutions. Therefore it may not be systematically applied to various examples.

Standard Melnikov method can also be applied to the sloshing problem. See similar discussion in [5] about the Melnikov approach to slowly varying pendulum studied in [3]. However, in order to conclude the existence of complicated dynamics in terms of symbolic sequences, one may need transverse intersection of stable and unstable manifolds, as well as time periodicity. The latter is required in order to reduce the ODE problem to a problem for diffeomorphisms so that one can apply the Poincaré-Birkhoff-Smale theorem about the existence of horseshoes near a transeverse homoclinic point. Our method does not need the transversality nor the periodicity. It can be applied to slowly varyinf systems with not necessarily time periodic (but merely oscillatory) forcing.

A Conley index theory for fast-slow systems

Consider the family of differential equations on $\mathbb{R}^n \times \mathbb{R}$ given by

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= \epsilon g(x, \lambda) \end{aligned} \tag{A.1}$$

where $f(x, \lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g(x, \lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are C^1 functions and $\epsilon \geq 0$. In this paper, we only consider the case where $g(x, \lambda) > 0$. A more general case is treated in [1]. The solutions to this equation generate a flow

$$\varphi^\epsilon : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}.$$

In the special case $\epsilon = 0$, (A.1) has a simpler form since λ is a constant. We can view λ as a parameter for the flows on $\mathbb{R}^n$, and for each l we define a flow ψ_λ on $\mathbb{R}^n$ by

$$(\psi_\lambda(t, x), \lambda) = \varphi^0(t, x, \lambda). \tag{A.2}$$

If we fix a range of values of λ, i.e., $\lambda \in \Lambda = [\lambda_0, \lambda_1]$, one can define a *parameterized flow*

$$\psi^\Lambda : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \times \Lambda$$

by $\psi^\Lambda(t, x, \lambda) := (\psi_\lambda(t, x), \lambda)$.

Consider for the moment an arbitrary flow γ defined on a locally compact metric space X, a compact set $N \subset X$ is an *isolating neighborhood* if

$$\mathrm{Inv}(N, \gamma) := \{x \in X \mid \gamma(\mathbb{R}, x) \subset N\} \subset \mathrm{Int} N.$$

If $S = \mathrm{Inv}(N, \gamma)$ for some isolating neighborhood N, then S is referred to as an *isolated invariant set*. The Conley index is an index of isolating neighborhoods with the property that if $\mathrm{Inv}(N, \gamma) = \mathrm{Inv}(N', \gamma)$ then the Conley index of N equals the Conley index of N'. In this way one may, also, view the Conley index as an index of isolated invariant sets. We shall make use of the cohomological Conley index which is denoted by $CH^*(S)$ and is an Alexander-Spanier cohomology group.

As was mentioned earlier, given an isolating neighborhood its Conley index can be used to describe the dynamics of the associated isolated invariant set. In our case we will present theorems which can be used to prove the existence of heteroclinic orbits.

The first step is to find the appropriate isolating neighborhoods. This is done by choosing compact neighborhoods of the connecting orbits and segments of the branches of equilibria. Observe, however, that this cannot produce an isolating neighborhood under the singular flow φ^0. On the other hand, our interest is in the dynamics for $\epsilon > 0$. Therefore, it is only important that the constructed neighborhood isolate under φ^ϵ when $\epsilon > 0$.

The second step is to compute the Conley index of the isolating neighborhood for $\epsilon > 0$. For that purpose, we will define building blocks for constructing an isolating neighborhood. The segments around the branches of equilibria are the simplest to define. Let ψ_λ be as in (A.2).

Definition A.1 $\mathcal{T} \subset \mathbb{R}^n \times \mathbb{R}$ is a *tube* if:

(1) There exists an interval $[a, b]$ such that $\mathcal{T} \subset \mathbb{R}^n \times [a, b]$ and $\mathcal{T}$ is an isolating neighborhood for

$$\begin{aligned} \psi^{\mathcal{T}} : \mathbb{R} \times \mathbb{R}^n \times [a, b] &\to \mathbb{R}^n \times [a, b], \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda). \end{aligned}$$

(2) There exists $\delta(\mathcal{T}) = \pm 1$ such that for all $(x, \lambda) \in \mathcal{T}$ we have $\delta(\mathcal{T}) g(x, \lambda) > 0$.

We now turn to the neighborhoods of the connecting orbits and the non-trivial problem of how to relate the index information between the various tubes. The Conley index theory provides a variety of techniques for proving the existence of heteroclinic connections. We shall use the following. Recall that a *Morse decomposition*

$$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$$

of an isolated invariant set S is a finite collection of disjoint compact invariant subsets $M(p)$, called *Morse sets*, indexed by a partially ordered set $(\mathcal{P}, >)$, with the property that; if $x \in S \setminus \bigcup_{p \in P} M(p)$, then there exist $q > p$ such that the alpha limit set of x is contained in $M(q)$ and the omega limit set of x is contained in $M(p)$.

In the context of a parametrized flow $\psi^\Lambda : \mathbb{R} \times X \times \Lambda \to X \times \Lambda$, a Morse decomposition is said to *continue over* Λ if there is an isolated invariant set $S = \mathrm{Inv}(N, \psi^\Lambda)$ with a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$. Observe that if one defines

$$S_\lambda := S \cap (\mathbb{R}^n \times \{\lambda\}),$$

then S_λ is an isolated invariant set for ψ_λ. Similarly, $\{M_\lambda(p) \mid p \in (\mathcal{P}, >)\}$ is a Morse decomposition for S_λ. Since Morse sets are isolated invariant sets, $CH^*(M_\lambda(p))$ is defined. Furthermore, the index of each Morse set remains constant over Λ. Let $\lambda_0, \lambda_1 \in \Lambda$ and assume that

$$S_{\lambda_i} = \bigcup_{p \in \mathcal{P}} M_{\lambda_i}(p), \quad i = 0, 1.$$

Then, there exists a lower triangular (with respect to the order $>$) degree 0 isomorphism

$$T^{l_1, l_0} : \bigoplus_{p \in \mathcal{P}} CH^*(M_{l_1}(p)) \to \bigoplus_{p \in \mathcal{P}} CH^*(M_{l_0}(p))$$

called a *topological transition matrix* (see [7, 8]). Roughly, if the p, q off diagonal entry of T^{l_1, l_0} is non-zero, then for some parameter value $l \in (l_0, l_1)$ there exists a connecting orbit between $M_l(p)$ and $M_l(q)$. As will become clear later, these off diagonal entries play a crucial role in the desired computation of the Conley index.

In order to insure the existence of topological transition matrices in the abstract setting of the fast-slow systems, we introduce the following neighborhoods of the connecting orbits.

Definition A.2 A set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}$ is a *box* if:

(1) There exists an interval $[c, d]$ such that $\mathcal{B} \subset \mathbb{R}^n \times [c, d]$ and $\mathcal{B}$ is an isolating neighborhood for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{aligned} \psi^{\mathcal{B}} : \mathbb{R} \times \mathbb{R}^n \times [c, d] &\to \mathbb{R}^n \times [c, d], \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda). \end{aligned}$$

(2) Let $S(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}, \psi^{\mathcal{B}})$. There exists a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) := \{M(p, \mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as the admissible ordering. Let $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathbb{R} \times \{\lambda\})$, $S_\lambda(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}_\lambda, \psi_\lambda)$ and let $\{M_\lambda(p, \mathcal{B}) \mid p = 1, \ldots, P_\mathcal{B}\}$ be the corresponding Morse decomposition of $S_\lambda(\mathcal{B})$. Then

$$S_c(\mathcal{B}) := \bigcup_{p=1}^{P_\mathcal{B}} M_c(p, \mathcal{B}) \quad \text{and} \quad S_d(\mathcal{B}) := \bigcup_{p=1}^{P_\mathcal{B}} M_d(p, \mathcal{B}).$$

(3) There are isolating neighborhoods $V(p, \mathcal{B})$ for $M(p, \mathcal{B})$ such that

$$V(p, \mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad V(p, \mathcal{B}) \cap V(q, \mathcal{B}) = \emptyset$$

for $p \neq q$ and for every $\lambda \in [c, d]$

$$V_\lambda(p, \mathcal{B}) \subset \mathrm{Int}(\mathcal{B}_\lambda).$$

Furthermore, there are $\delta(p, \mathcal{B}) = \pm 1$, $p = 1, \ldots, P_\mathcal{B}$, such that

$$\delta(p, \mathcal{B}) g(x, \lambda) > 0 \qquad \text{for all } (x, \lambda) \in V(p, \mathcal{B}).$$

Notice that Definition A.2(2) implies that there are no connecting orbits between the Morse sets at the parameter values c and d, and by the construction, the sets $S_c(\mathcal{B})$ and $S_d(\mathcal{B})$ are related by continuation.

If one is attempting to prove the existence of heteroclinic orbits, an additional type of neighborhood which surrounds the critical points for the perturbed system is necessary.

Definition A.3 A set $\mathcal{C}(R)$ $(\mathcal{C}(A))$ is a *repelling (attracting) cap* if:

(1) There exists an interval $[e, f]$ such that $\mathcal{C} \subset \mathbb{R}^n \times [e, f]$ and $\mathcal{C}$ is an isolating neighborhood for

$$\begin{aligned} \psi^\mathcal{C} : \mathbb{R} \times \mathbb{R}^n \times [e, f] &\to \mathbb{R}^n \times [e, f] \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda) \end{aligned}$$

(2)

$$\begin{aligned} x \in \mathcal{C}_e(R) &\quad \Rightarrow \quad g(x, e) < 0 \\ x \in \mathcal{C}_f(R) &\quad \Rightarrow \quad g(x, f) > 0 \\ x \in \mathcal{C}_e(A) &\quad \Rightarrow \quad g(x, e) > 0 \\ x \in \mathcal{C}_f(A) &\quad \Rightarrow \quad g(x, f) < 0, \end{aligned}$$

where $\mathcal{C}_\lambda(R) := \mathcal{C}(R) \cap \{\lambda\}$ and $\mathcal{C}_\lambda(A) := \mathcal{C}(A) \cap \{\lambda\}$.

Finally, in order to construct a global isolating neighborhood, these boxes, tubes, and caps must be related in a consistent manner. The primary requirement is that the tubes and boxes overlap at the appropriate Morse sets. To simplify the notation we let $P_i = P_{\mathcal{B}_i}$ and $M(p, i) := M(p, \mathcal{B}(i))$.

Definition A.4 A *tubes, boxes and caps collection* (TBC collection) is a collection of tubes $\{\mathcal{T}(i)\,|\,i=1,\dots,I+1\}$, boxes $\{\mathcal{B}(i)\,|\,i=1,\dots,I\}$, and caps $\mathcal{C}(R)$ and $\mathcal{C}(A)$ such that:

(1) for $i=1,\dots,I$,

  (a) $\mathcal{T}(i)\cap(\mathbb{R}\times[c_i,d_i])\subset V(1,\mathcal{B}(i))$ and $\mathcal{T}(i)\cap\mathcal{B}(i)$ isolates $M(1,i)$.

  (b) $\mathcal{T}(i+1)\cap(\mathbb{R}\times[c_i,d_i])\subset V(P_i,\mathcal{B}(i))$ and $\mathcal{T}(i+1)\cap\mathcal{B}(i)$ isolates $M(P_i,i)$.

(2) for $i=1,\dots,I$, either

$$\delta(\mathcal{T}(i+1))>0 \text{ and } \delta(P_i,\mathcal{B}(i))>0 \text{ in which case } b_{i+1}=d_i$$

or

$$\delta(\mathcal{T}(i+1))<0 \text{ and } \delta(P_i,\mathcal{B}(i))<0 \text{ in which case } a_{i+1}=c_i$$

where a, b, c, and d are as in Definitions A.1 and A.2.

(3) for $i=1,\dots,I$, either

$$\delta(\mathcal{T}(i))>0 \text{ and } \delta(1,\mathcal{B}(i))>0 \text{ in which case } a_i=c_i$$

or

$$\delta(\mathcal{T}(i))<0 \text{ and } \delta(1,\mathcal{B}(i))<0 \text{ in which case } b_i=d_i$$

where a, b, c, and d are as in Definitions A.1 and A.2.

(4) If $i\neq j$, then $\mathcal{B}(i)\cap\mathcal{B}(j)=\emptyset$.

(5) $\mathcal{C}(R)\cap\mathcal{T}(I+1)\neq\emptyset$ and $\mathcal{C}(A)\cap\mathcal{T}(1)\neq\emptyset$. Furthermore,

$$\begin{aligned}\mathcal{C}(R)\cap\mathcal{T}(I+1)\cap(\mathbb{R}^n\times\{\lambda\})\neq\emptyset \quad&\Rightarrow\quad \mathcal{C}_\lambda(R)=\mathcal{T}_\lambda(I+1),\\ \mathcal{C}(A)\cap\mathcal{T}(1)\cap(\mathbb{R}^n\times\{\lambda\})\neq\emptyset \quad&\Rightarrow\quad \mathcal{C}_\lambda(A)=\mathcal{T}_\lambda(1).\end{aligned}$$

Given a TBC collection, let

$$T^i:\bigoplus_{p=1}^{\mathcal{P}_i}CH^*(M_{d_i}(p,i))\to\bigoplus_{p=1}^{\mathcal{P}_i}CH^*(M_{c_i}(p,i))$$

denote the transition matrix associated with the box $\mathcal{B}(i)$ and let

$$T^i(P_i,1):CH^*(M_{d_i}(1,i))\to CH^*(M_{c_i}(P_i,i)) \tag{A.3}$$

denote its corresponding entry. Again, having fixed the TBC collection, we define a map

$$\Theta:=\Theta(I)=T^I(P_I,1)\circ T^{I-1}(P_{I-1},1)\circ\ldots\circ T^2(P_2,1)\circ T^1(P_1,1). \tag{A.4}$$

This definition makes sense since the continuation of the Conley index allows for a natural identification between these spaces.

The following result can be used to find heteroclinic orbits. We begin with a concept concerning the dynamics within the isolating neighborhood.

The simplest non-trivial Morse decomposition of an isolated invariant set S consists of two Morse sets $M(1)$ and $M(0)$ with an admissible ordering $1>0$. In this case, $M(0)$ is called an *attractor* in S and $M(1)$ a *repeller*. Together, the pair $(M(0),M(1))$ is referred to as an *attractor repeller pair decomposition* of S.

Theorem A.5 *Let* $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$, $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ *and* $\mathcal{C}(R), \mathcal{C}(A)$ *be a TBC collection. Let*

$$\mathcal{N} := \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I+1} \mathcal{T}(i) \cup \mathcal{C}(R) \cup \mathcal{C}(A).$$

Then, for $\epsilon > 0$ *sufficiently small,*

(1) $\mathcal{N}$ *is an isolating neighborhood for* φ^ϵ;

(2) $(\mathrm{Inv}(\mathcal{C}(R), \varphi^\epsilon), \mathrm{Inv}(\mathcal{C}(A), \varphi^\epsilon))$ *is an attractor-repeller pair for* $\mathrm{Inv}(\mathcal{N}, \varphi^\epsilon)$;

(3) If $\Theta \neq 0$, *then*

$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\epsilon)) \not\cong CH^*(\mathrm{Inv}(\mathcal{C}(A), \varphi^\epsilon)) \oplus CH^*(\mathrm{Inv}(\mathcal{C}(R), \varphi^\epsilon)).$$

In particular, for all sufficiently small $\epsilon > 0$, *there is a connecting orbit from* $\mathrm{Inv}(\mathcal{C}(R), \varphi^\epsilon)$ *to* $\mathrm{Inv}(\mathcal{C}(A), \varphi^\epsilon)$ *in* $\mathcal{N}$ *under the flow* φ^ϵ.

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