Existence of infinitely many connecting orbits in a singularly perturbed ordinary differential equation*

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Abstract

We consider a one-parameter family of two-dimensional ordinary differential equations with a slow parameter drift. Our equation assumes that when there is no parameter drift, there are two invariant curves consisting of equilibria, one of which is normally hyperbolic and whose stable and unstable manifolds intersect transversely. The slow parameter drift is introduced in a way that it creates two hyperbolic equilibria in the invariant normally hyperbolic curve that is persistent under perturbation. In this situation, we prove that the number of distinct orbits which connects these two equilibria changes from finite to infinite depending on the direction of the slow parameter drift. The proof uses the Conley index theory. The relation to a singular boundary value problem studied by W. Kath is, also, discussed.

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1 Introduction

In this paper, we consider a one-parameter family of two-dimensional ordinary differential equations with a slow parameter drift, which is given by the following form:

$$\begin{aligned} \dot{u} &= f(u, \lambda, \epsilon), \\ \dot{\lambda} &= \epsilon g(\lambda), \end{aligned} \qquad u \in \mathbb{R}^2,\ \epsilon \in \mathbb{R},\ \lambda \in [-1, 1], \tag{1}$$

where $g : [-1, 1] \to \mathbb{R}$ is C^1 and satisfies

$$g(\lambda) > 0 \text{ for } -1 < \lambda < 1, \qquad \text{and} \qquad g(\pm 1) = 0.$$

Such a family can be viewed as a singularly perturbed autonomous ordinary differential equation (a vector field) or its time-rescaled version. The purpose of this paper is to present a topological method which allows one to prove existence of infinitely many connecting orbits in this type of equations under some conditions that are detailed below. Before giving a precise statement of the result, we shall explain some motivation and background of the study.

This work is stimulated by a paper due to Kath[6], who studied a slowly varying singular boundary value problem of the form:

$$\mu^2 \frac{d^2 y}{dx^2} + f(y, x) = 0, \quad y(x = \pm 1) = 0,$$

where $\mu > 0$ is a small perturbation parameter and the slowly varying potential $F(y, x) = \int f(y, x) dy$ is assumed (as a typical case) to have one minimum and one maximum when $x = 0$, so that the corresponding autonomous problem

$$\mu^2 \frac{d^2 y}{dx^2} + f(y, 0) = 0$$

possesses a center and a saddle with a homoclinic orbit in its phase plane. Kath approached this problem from qualitative point of view by working in the phase plane instead of using analytical techniques such as matched asymptotic expansions, and he observed that, in some cases, the stable and unstable manifolds of the saddle (which persist under the presence of slowly varying variable x in the nonlinear term $f(y, x)$) can have infinitely many (transverse) intersections in the phase plane. The existence of infinitely many intersections that correspond to infinitely many distinct solutions to the problem heavily depends on the mutual position of the stable and unstable manifolds of the saddle, which were measured by a Melnikov-like function or the "energy difference" in [6] (notice that when $x = 0$ the problem gives a Hamiltonian equation.) See [6] for more detail.

Unfortunately, the argument given in [6], although very interesting, did not allow us to understand the essential mechanism for creating infinitely many such different solutions to this problem. Motivated by [6], we shall try to give a complete mathematical proof for this type of result. In this paper, we consider an initial value problem instead of a boundary value problem, as Kath essentially did the same by using the phase plane analysis, and hence we view the parameter μ in the above equation as one of the phase variables and similarly $\mu = \pm\infty$ as its equilibrium points. In this way, the problem becomes one of finding infinitely many orbits that connect two equilibrium points corresponding to $\mu = \pm\infty$. Our idea is the use of the Conley index theory which provides us with a topological method for detecting connecting orbits. In order to focus only on the essential features of the problem and since, as explained in [6], the mutual position of the stable and unstable manifolds should be the most important feature in this problem, we shall modify the equation by removing the slowly varying variable and replacing it by a friction term which makes the manner of splitting of the stable and unstable manifolds less degenerate. This modification can be justified by using a result in [15].

A typical form of the problem could therefore be given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda y - x(x-a), \qquad x, y, \lambda, a, \epsilon \in \mathbb{R}, \\ \dot{\lambda} &= \epsilon(1-\lambda^2), \end{aligned} \tag{2}$$

where λy is the friction term and the equilibrium points are transformed to $(x, y, \lambda) = (0, 0, \pm 1)$. When $\epsilon \neq 0$, there are exactly four equilibria, $B^{\pm} = (0, 0, \pm 1)$ and $A^{\pm} = (a, 0, \pm 1)$. Our main result of this paper will show that for $\epsilon < 0$ there exist exactly two distinct heteroclinic orbits $B^+ \to B^-$, i.e. solutions $(x(t), y(t), \lambda(t))$ to (2) such that

$$\begin{aligned} \lim_{t\to\infty}(x(t), y(t), \lambda(t)) &= (0, 0, -1) = B^- \\ \lim_{t\to-\infty}(x(t), y(t), \lambda(t)) &= (0, 0, 1) = B^+ \end{aligned} \tag{3}$$

while for $\epsilon > 0$ there exists infinitely many distinct heteroclinic orbits $B^- \to B^+$. To describe the assumptions and to state the results rigorously requires some notation which we now introduce.

First, for technical reasons we wish to extend the set of parameter values λ at least to a set $\Lambda = [-1-\mu, 1+\mu]$ for some arbitrary $\mu > 0$. In some applications it is natural to assume that $g(\lambda)$ and $f(\cdot, \lambda, \cdot)$ are defined over $\mathbb{R}$, and hence, Λ. However, in the case of singular boundary value problems one is lead to consider equations of the form

$$\begin{aligned} \dot{u} &= f(u, \xi, \epsilon), \\ \dot{\xi} &= \epsilon, \end{aligned} \qquad u \in \mathbb{R}^2,\ \xi \in \mathbb{R},\ \epsilon \in \mathbb{R}, \tag{4}$$

where it is assumed that

$$\lim_{\xi \to \pm\infty} f(u, \xi, \epsilon) = F^{\pm}(u, \epsilon)$$

is well defined. In this case one can use the change of variables

$$\lambda = \frac{2}{\pi} \tan^{-1} \xi$$

to transfer (4) to

$$\begin{aligned} \dot{u} &= f(u, \tan \frac{\pi}{2}\lambda, \epsilon) \\ \dot{\lambda} &= \epsilon \frac{2}{\pi} \cos^2(\frac{\pi}{2}\lambda) \end{aligned}$$

which is in the form of (1). Of course, in this case the natural values of λ are in the interval $[-1, 1]$. Thus, in this setting, to extend to $\lambda \in \Lambda$ one can define

$$f(u, \lambda, \epsilon) = \begin{cases} F^+(u, \epsilon) & \text{if } 1 \le \lambda \le 1 + \mu \\ f(u, \lambda, \epsilon) & \text{if } -1 < \lambda < 1 \\ F^-(u, \epsilon) & \text{if } -1 - \mu \le \lambda \le -1 \end{cases}$$

and choose g such that for $\lambda \neq \pm 1$, $(1 - \lambda^2) g(\lambda) > 0$. This results in the system which will be studied

$$\begin{aligned} \dot{u} &= f(u, \lambda, \epsilon), \\ \dot{\lambda} &= \epsilon g(\lambda), \end{aligned} \qquad u \in \mathbb{R}^2,\ \lambda \in \mathbb{R},\ \epsilon \in \mathbb{R}, \tag{5}$$

under the assumption that $(1-\lambda^2)g(\lambda) > 0$ when $\lambda \neq \pm 1$.

Observe that when $\epsilon = 0$, the system reduces to a continuous 1-parameter family of equations

$$\dot{u} = f(u, \lambda, 0). \tag{6}$$

Let $\varphi^\lambda : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the flow generated by (6). Let

$$\Phi : \mathbb{R} \times \mathbb{R}^2 \times \Lambda \to \mathbb{R}^2 \times \Lambda$$

be the *parameterized flow* over the parameter space Λ, i.e.

$$\Phi(t, u, \lambda) = (\varphi^\lambda(t, u), \lambda).$$

Recall that a compact set N is an *isolating neighborhood* for a flow ψ if the maximal invariant set of N is contained in the interior of N, i.e.,

$$\mathrm{Inv}(N, \psi) := \{u \mid \psi(\mathbb{R}, u) \subset N\} \subset \mathrm{int} N.$$

We shall make the following assumptions.

A1 *There exists an isolating neighborhood N of Φ such that N is homeomorphic to $[0,1]^3$ and $N^\lambda := N \cap (\mathbb{R}^2 \times \{\lambda\})$ is homeomorphic to $[0,1]^2$ for all $\lambda \in \Lambda$.*

As will be seen, N serves to define the region of phase space on which the dynamics of interest occurs. In particular, any heteroclinic orbits of $\Phi_\epsilon : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, the flow generated by (5), which do not lie entirely in N will be ignored. In [10] it is shown that if N is an isolating neighborhood for Φ then there exists $\bar{\epsilon} > 0$ such that for $\epsilon \in [-\bar{\epsilon}, \bar{\epsilon}]$, N is an isolating neighborhood of Φ_ϵ.

A2 *There exists $A \in \mathbb{R}^2$ and a continuous function $B : \Lambda \times [-\bar{\epsilon}, \bar{\epsilon}] \to \mathbb{R}^2$ such that*

$$f(A, \lambda, \epsilon) = 0, \qquad f(B(\lambda, \epsilon), \lambda, \epsilon) = 0$$

for all $\lambda \in \Lambda$ and $|\epsilon| \leq \bar{\epsilon}$. Furthermore, for each $\lambda \in [-1, 1]$, A and $B(\lambda, 0)$ are the only equilibria of φ^λ in N^λ.

A3 *Let $\alpha_i(\lambda)$ and $\beta_i(\lambda)$, $i = 1, 2$, denote the eigenvalues of $Df(A, \lambda, 0)$ and $Df(B(\lambda, 0), \lambda, 0)$ respectively. For $-1 \leq \lambda \leq 1$,*

$$\beta_1(\lambda) < 0 < \beta_2(\lambda), \quad \lambda \in [-1, 1]$$

$$\lambda Re\, \alpha_i(\lambda) > 0 \quad for \quad \lambda = \pm 1$$

and

$$Im\, \alpha_i(\lambda) \neq 0 \quad \text{when} \quad Re\, \alpha_i(\lambda) = 0.$$

Observe that this forces $B(\lambda, 0)$ to be a saddle point while A is a repelling (attracting) fixed point for φ^λ when $\lambda > 0$ ($\lambda < 0$). The assumptions **A2** and **A3** are local in nature. The next two assumptions contain global information.

A4 Inv(N^1, φ^1) *consists of the critical points A and $B(1,0)$ and a unique heteroclinic orbit from A to $B(1,0)$.* Inv(N^{-1}, φ^{-1}) *consists of the critical points A and $B(-1,0)$ and a unique heteroclinic orbit from $B(-1,0)$ to A.*

Let $B_\epsilon^+ = (B(1,\epsilon), 1) \in N$ and $B_\epsilon^- = (B(-1,\epsilon), -1) \in N$. Observe that $B_\epsilon^\pm$ are equilibria of Φ_ϵ. Let the set of connecting orbits from B_ϵ^+ to B_ϵ^- in N under Φ_ϵ be denoted by $C(B_\epsilon^+, B_\epsilon^-; N, \Phi_\epsilon)$, i.e.,

$$\begin{aligned} C(B_\epsilon^+, B_\epsilon^-; N, \Phi_\epsilon) := \{&(u,\lambda) \in N \mid \Phi_\epsilon(\mathbb{R}, u, \lambda) \subset N, \\ &\lim_{t\to\infty} \Phi_\epsilon(t,u,\lambda) = B_\epsilon^-, \lim_{t\to-\infty} \Phi_\epsilon(t,u,\lambda) = B_\epsilon^+\}. \end{aligned}$$

$C(B_\epsilon^-, B_\epsilon^+; N, \Phi_\epsilon)$ is defined similarly. Let $\#C(B_\epsilon^-, B_\epsilon^+; N, \Phi_\epsilon)$ denote the number of components of $C(B_\epsilon^-, B_\epsilon^+; N, \Phi_\epsilon)$.

The following subset of N will be used to obtain a useful collection of isolated connections. Let

$$D = N \backslash (\{A\} \times [-1-\mu, 1+\mu]).$$

Observe that $\{A\} \times [-1,1]$ is an invariant set for Φ_ε. Therefore if γ_ε denotes a $B_\varepsilon^+ \to B_\varepsilon^-$ connecting orbit, then $\gamma_\varepsilon \cap \{A\} \times [-1,1] = \emptyset$ and $\gamma_\varepsilon \subset D$. Therefore, $C(B_\varepsilon^+, B_\varepsilon^-; D, \Phi_\varepsilon) = C(B_\varepsilon^+, B_\varepsilon^-; N, \Phi_\varepsilon)$.

By **A1**, D is homotopic to S^1, and hence $\pi_1(D) \approx \pi_1(S^1) \approx \mathbb{Z}$ where π_1 denotes the fundamental group. This suggests the following definition.

Definition 1.1 Let $\bar{B}_\varepsilon = \bigcup_{-1\le\lambda\le1} B(\lambda, 0) \cup \bigcup_{0\le|\mu|\le|\varepsilon|} B(\pm1, \mu)$. Let γ_ε denote a $B_\varepsilon^+ \to B_\varepsilon^-$ connecting orbit. Then $\gamma_\varepsilon \cup \bar{B}_\varepsilon$ generates an element of $\pi_1(D)$. The *winding number* of γ_ε is defined to be

$$\theta(\gamma_\varepsilon) := [\gamma_\varepsilon \cup \bar{B}_\varepsilon] \in \mathbb{Z}.$$

Using the notion of the winding number, we make the final assumption as follows:

A5 *Let γ_{ε_n} be a $B^+_{\varepsilon_n} \to B^-_{\varepsilon_n}$ connecting orbit with $\varepsilon_n < 0$ and $\theta(\gamma_{\varepsilon_n}) \leq 0$. If γ_{ε_n} converges to a set Γ in the Hausdorff topology on compact sets as $\varepsilon_n \to 0$, then $\Gamma = \bar{B} := \bar{B}_0$.*

This assumption can be verified by looking at the invariant set of the parametrized flow. For instance, it is easy to see that the model equation (2) satisfies **A5**. In fact, since the limit set Γ is a connected invariant set of the parametrized flow which contains B^+ and B^-, it must consist of the line segment $\bar{B}$ and, possibly, some of the homoclinic or periodic orbit of the Hamiltonian flow

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(x-a), \end{aligned}$$

in the plane given by $\lambda = 0$. But if it contains a homoclinic or periodic orbit, it must turn at least once around the set $\{A\} \times \mathbb{R}$, and hence the winding number cannot be less than or equal to 0, since all the periodic and the homoclinic orbits rotate in the same direction.

The goal of this paper is to prove the following theorem.

Theorem 1.2 *Given assumptions* **A1-A5** *and $|\epsilon| \leq \bar{\epsilon}$:*

(a) if $\epsilon < 0$, then $\#C(B^+_\epsilon, B^-_\epsilon; N, \Phi_\epsilon)$ is bounded and ≥ 2;

(b) if $\epsilon > 0$, then $\#C(B^-_\epsilon, B^+_\epsilon; N, \Phi_\epsilon) = \infty$.

Moreover, these connecting orbits are distinguished by the winding numbers.

Let us give a heuristic proof for the theorem in order to clarify the geometric idea behind it and to motivate the rigorous argument presented in the subsequent sections. Our goal is to find connecting orbits between B^-_ε and B^+_ε for sufficiently small $\varepsilon \neq 0$. Note that the set of equilibria $B(\lambda, 0)$ for the fast dynamics forms an invariant curve in the parameterized flow and from **A3** it is normally hyperbolic, hence it has a stable and an unstable manifold. The assumptions **A3** and **A4** imply that these manifolds wrap around the line $\{A\} \times [-1-\mu, 1+\mu]$ and let us assume that they intersect transversely as indicated in Figure 1. It then follows that there exists at least one connection between B^-_ε and B^+_ε besides the trivial connection.

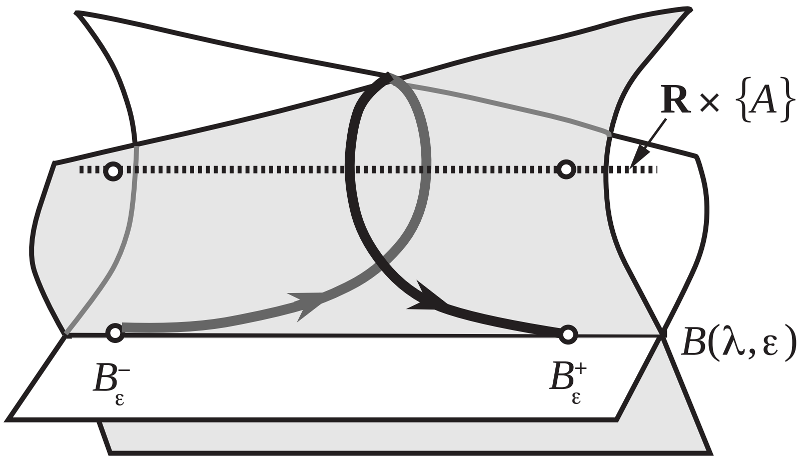

Figure 1: The transverse intersection of the stable and unstable manifolds for the curve of hyperbolic fixed points $B(\lambda, 0)$.

For our purpose, it is easier to work on the universal covering space. To be more precise, we consider the set $\mathbb{R}^2 \setminus \{A\} \times [-1-\mu, 1+\mu]$ and take its universal covering. The picture we obtain by lifting Figure 1 to the universal covering becomes either like Figure 2 or like Figure 3, depending on the direction of the slow flow which is determined by the sign of ε. In particular, $B^{\pm}_{\varepsilon,n}$ denotes the equilibria given by the lift of $B^{\pm}_{\varepsilon}$, respectively. The direction of the flow is chosen in such a way that one turn around the set $\{A\} \times [-1-\mu, 1+\mu]$ in the base space corresponds to going one floor up in the covering space. For instance, a connection from B^-_ε to B^+_ε in the base space corresponds to a connection from $B^-_{\varepsilon,0}$ to $B^+_{\varepsilon,k}$ for some $k \geq 0$ in the universal covering space. Note that the connection between $B^-_{\varepsilon,k}$ and $B^+_{\varepsilon,k}$ for any $k \in \mathbb{Z}$ indicates the lift of the invariant curve close to $B(\lambda, 0)$ for all $\lambda \in [-1-\mu, 1+\mu]$.

Let us first consider the case $\varepsilon < 0$ corresponding to Figure 2. In this case, we are looking for $B^+_{\varepsilon,0} \to B^-_{\varepsilon,k}$ connections for $k > 0$. Clearly there exists one such connection since there exists at least one connection from B^+_ε to B^-_ε in the base dynamics due to the transverse intersection of the stable and unstable manifolds. For simplicity we think of this connection in the universal covering space as a $B^+_{\varepsilon,0} \to B^-_{\varepsilon,1}$ connection. By the deck transformation, we also have a $B^+_{\varepsilon,k} \to B^-_{\varepsilon,k+1}$ connection for each $k \in \mathbb{Z}$.

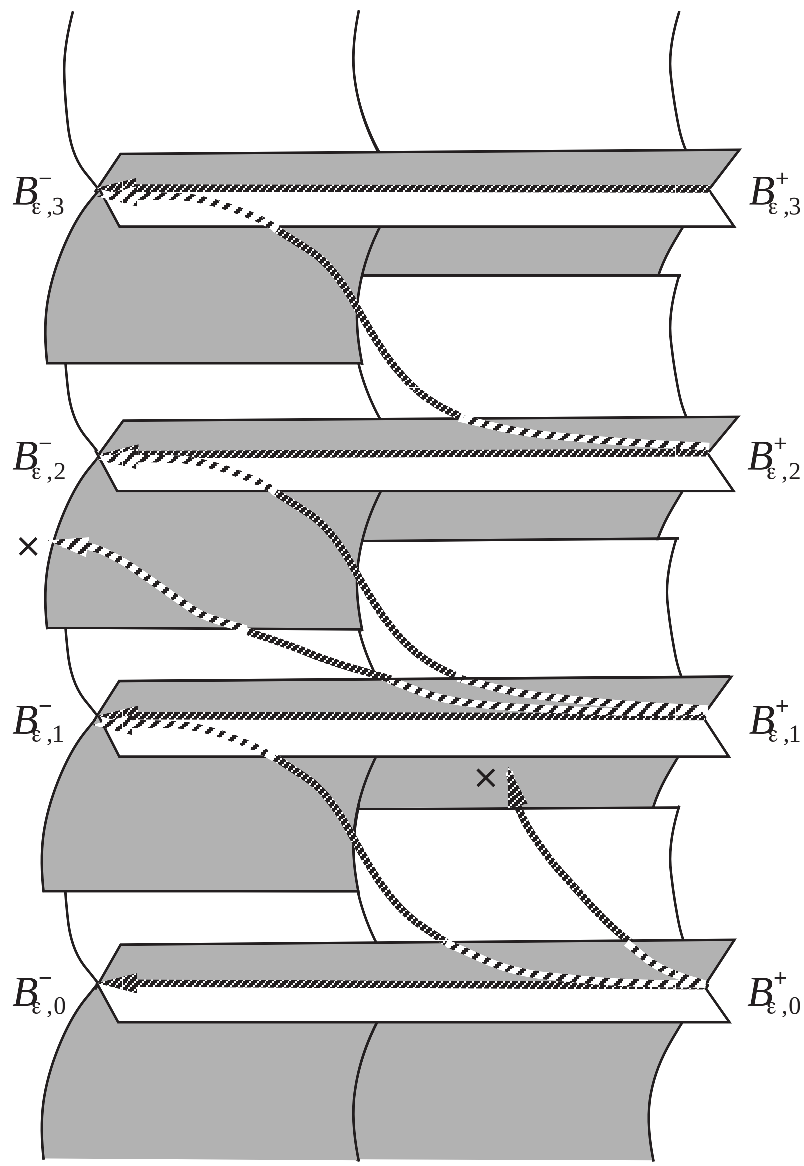

Figure 2: The intersections of the stable and unstable manifolds of the points $B^{\pm}_{\varepsilon}$ for $\varepsilon < 0$.

Besides these connections as well as trivial $B^+_{\varepsilon,k} \to B^-_{\varepsilon,k}$ connections, we hope to convince the reader that there is no way of finding other connections in Figure 2. Suppose there could exist a $B^+_{\varepsilon,0} \to B^-_{\varepsilon,2}$ connection. It then must lie in the unstable manifold of $B^+_{\varepsilon,0}$ and the stable manifold of $B^-_{\varepsilon,2}$, and hence the connecting orbit must go two floors up in the figure. However there already exists a $B^+_{\varepsilon,0} \to B^-_{\varepsilon,1}$ connection. If the desired $B^+_{\varepsilon,0} \to B^-_{\varepsilon,2}$ connection goes lower than the $B^+_{\varepsilon,0} \to B^-_{\varepsilon,1}$ connection in the unstable manifold of $B^+_{\varepsilon,0}$, then it fails to get on and goes underneath the stable manifold of $B^+_{\varepsilon,1}$ and hence it cannot reach $B^-_{\varepsilon,2}$. If on the other hand, the $B^+_{\varepsilon,0} \to B^-_{\varepsilon,2}$ connection goes above the $B^+_{\varepsilon,0} \to B^-_{\varepsilon,1}$ connection, then it runs over the stable manifold of $B^-_{\varepsilon,1}$ and hence it again cannot go across to $B^-_{\varepsilon,2}$. Therefore in either case, there could not exist $B^+_{\varepsilon,0} \to B^-_{\varepsilon,2}$ connections. Similarly one can imagine that there also exist no $B^+_{\varepsilon,0} \to B^-_{\varepsilon,k}$ connections for any $k \geq 2$.

If the direction of the slow dynamics changes, namely if $\varepsilon > 0$, then the situation changes completely as indicated in Figure 3. In this case there exist $B^-_{\varepsilon,0} \to B^+_{\varepsilon,k}$ connections for any $k \geq 0$. Indeed, if you go slightly higher than the a priori existing $B^-_{\varepsilon,0} \to B^+_{\varepsilon,1}$ connection in the unstable manifold of $B^-_{\varepsilon,0}$, then you first go underneath the stable manifold of $B^+_{\varepsilon,1}$, cross the trivial $B^-_{\varepsilon,1} \to B^+_{\varepsilon,1}$ connection, and finally are able to get on the stable manifold of $B^+_{\varepsilon,2}$. Such a connection can clearly exist between $B^-_{\varepsilon,0}$ and $B^+_{\varepsilon,k}$ for any $k \geq 2$, and therefore, going back to the base space, we have infinitely many connecting orbits as desired.

This argument is meant to give a geometric basis for Theorem 1.2. However, our proof is algebraic and makes use of the Conley index theory rather than explicit assumptions concerning the transversal intersections of stable and unstable manifolds. In particular, we compute the Conley indices of the connecting orbits of Φ_ε when $\varepsilon < 0$, directly, and then use this information to compute the Conley index for the connecting orbits when $\varepsilon > 0$. It is worth remarking that in the context of the Conley index theory this is a singular transformation. Recall that the Conley index is defined for isolated invariant sets. However, as one passes from $\varepsilon < 0$ to $\varepsilon > 0$, one passes through the parameterized flow Φ_0 at which point all relevant isolation is lost. The key to our being able to make this transformation involves a new result due to C. McCord and the second author [9] which allows us to associate algebraic properties to the transition matrices used in our computations. This result and the relevant background for transition matrices is presented in the next section.

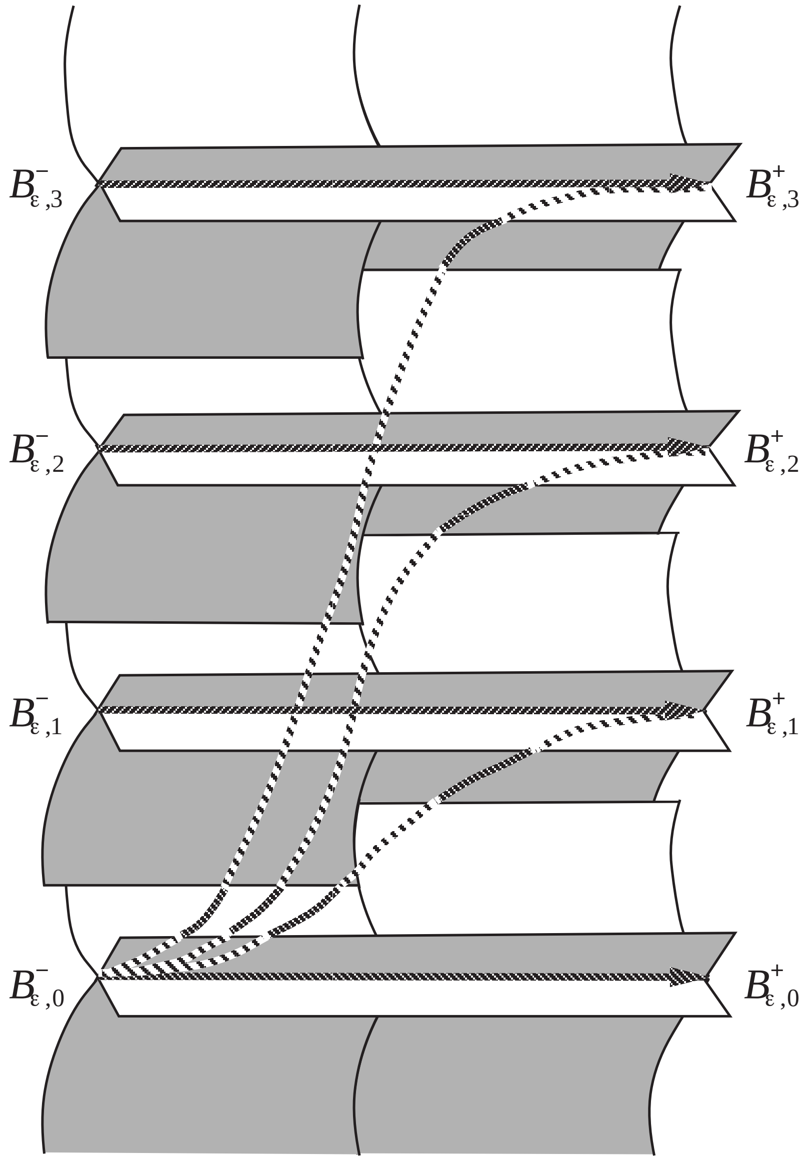

Figure 3: The intersections of the stable and unstable manifolds of the points $B_\varepsilon^\pm$ for $\varepsilon > 0$.

In Section 3, Theorem 1.2(a) is proven. This proof follows directly from the algebraic properties of Conley's connection matrix. Part (b) of Theorem 1.2 is proven in Section 4 and makes use of the above mentioned result concerning transition matrices.

2 Conley Index Theory

As was mentioned in the introduction our proof of Theorem 1.2 makes use of the Conley index theory. Thus, we begin with a brief review of the relevent portions of the theory. However, the primary purpose of this section is to establish the notation and the reader is referred to [1, 3, 12, 14, 17, 18, 19] for details.

Recall that a compact set $N \subset X$ is an *isolating neighborhood* for a flow $\psi : \mathbb{R} \times X \to X$ if

$$S = \mathrm{Inv}(N, \psi) := \{x \in N \mid \psi(\mathbb{R}, x) \subset N\} \subset \mathrm{int} N.$$

In this case the maximal invariant set S of N is called an *isolated invariant set*.

Given an isolated invariant set S, one can find a pair of compact sets (N, L) which form a regular index pair [18], in which case the (homological) *Conley index* of S is given by

$$CH_*(S) \approx H_*(N, L).$$

Remark 2.1 To simplify the calculations we shall always use homology with $\mathbb{Z}_2$ coefficients, i.e.

$$CH_*(S) \approx H_*(N, L; \mathbb{Z}_2).$$

There are various ways of decomposing an isolated invariant set which are natural with respect to the index theory. The most important of these is the following. A pair of compact invariant subsets (A, R) of an isolated invariant set, S, forms an *attractor-repeller pair* decomposition of S if

1. A is an *attractor* in S, i.e. there is a neighborhood U of A such that $\omega(U \cap S) = A$.

2. R is the *dual repeller* to A in S, i.e. $R = S \setminus \{x \mid \omega(x) \subset A\}$

where α and ω denote the alpha and omega limit sets. The set of *connections* from R to A within S is defined by

$$C(R, A; S) = \{x \in S \mid \alpha(x) \subset R, \omega(x) \subset A\}.$$

An obvious, but crucially important observation, is that

$$S = R \cup C(R, A; S) \cup A.$$

When there is no confusion concerning the invariant set S we shall simplify the notation and merely write $C(R, A)$ to denote the set of connecting orbits.

The natural generalization of an attractor-repeller pair decomposition is a Morse decomposition. To be precise, a *Morse decomposition* of an isolated invariant set S is a collection of disjoint compact invariant subsets of S,

$$\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\}$$

indexed by a finite set $\mathcal{P}$, for which one can find a partial order $>$ such that if $x \notin S \cup \left(\bigcup_{p \in \mathcal{P}} M(p)\right)$, then there exists $p, q \in \mathcal{P}$ such that $p > q$, $\alpha(x) \subset M(p)$, and $\omega(x) \subset M(q)$. Any partial order which satisfies the above condition is *admissible*. These individual invariant subsets, $M(p)$ are called *Morse sets*, and the remaining portion, $S \setminus \cup M(p)$, is referred to as the set of *connecting orbits*. In particular, given two Morse sets $M(p)$ and $M(q)$, the set of connecting orbits from $M(p)$ to $M(q)$ is denoted, as before, by

$$C(M(p), M(q); S) := \{x \in S \mid \omega(x) \subset M(q),\ \alpha(x) \subset M(p)\}.$$

If $\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\}$ is a Morse decomposition of S, then each $M(p)$ is an isolated invariant set. S contains other isolated invariant sets, some of which can be produced by the partial order on $\mathcal{P}$ as follows. A subset $I \subset \mathcal{P}$ is an *interval* in $\mathcal{P}$ if $r \in I$ whenever $p < r < q$ and $p, q \in I$. Disjoint intervals are *adjacent* if $IJ = I \cup J$ is also an interval (i.e. if no element of $\mathcal{P}$ lies "between" I and J). If I is an interval, let

$$M(I) := \left(\bigcup_{i \in I} M(i)\right) \bigcup \left(\bigcup_{i,j \in I} C(M(j), M(i))\right).$$

A *regular index triple* for an attractor–repeller pair (A, R) in S is a triple of compact spaces (N_2, N_1, N_0) such that (N_2, N_0) is a regular index pair for

S, (N_2, N_1) is a regular index pair for R and (N_1, N_0) is a regular index pair for A. Since the Conley index of an isolated invariant set is defined in terms of regular index pairs, regular index triples provides for the following relation between the Conley indices of the invariant sets. The inclusion induced exact sequence

$$\xrightarrow{\partial} H_n(N_1, N_0) \to H_n(N_2, N_0) \to H_n(N_2, N_1) \xrightarrow{\partial} H_{n-1}(N_1 N_0) \to$$

can be re-written as an exact sequence

$$\xrightarrow{\partial(R,A)} CH_n(R) \to CH_n(S) \to CH_n(A) \xrightarrow{\partial(R,A)} CH_{n-1}(R) \to$$

which is referred to as the homology *attractor-repeller pair sequence*. The boundary map $\partial(R, A)$ is called the *connecting homomorphism*, as $\partial(R, A) \neq 0$ implies that connections between R and A exist.

Just as one can decompose isolated invariant sets, one can decompose sets of connecting orbits. Given an attractor repeller pair decomposition (A, R) of an isolated invariant set S and an indexing set J a *separation* of $C(R, A)$ is a collection $\{C_j(R, A) \mid j \in J\}$ of open invariant subsets of $C(R, A)$ such that

$$C(R, A) = \bigcup_{j \in J} C_j(R, A).$$

It is easy to check that this forces J to be a finite set. Furthermore, if N is an isolating neighborhood of S, then for each $j \in J$ there exists, $N_j \subset N$ such that N_j is an isolating neighborhood for $A \cup R \cup C_j(R, A)$. For this reason, the sets $C_j(R, A)$ are called sets of *isolated connections*. Let

$$S_j = \mathrm{Inv} N_j,$$

then (A, R) is an attractor repeller pair decomposition of S_j. This of course, implies that there is an associated index triple (N_{j2}, N_{j1}, N_{j0}) and the corresponding connection map $\partial(R, A; j)$.

Theorem 2.2 *(C. McCord, [7, Theorem 2.5]) For any separation of* $C(R, A)$,

$$\partial(R, A) = \sum_{j \in J} \partial(R, A; j).$$

Connection matrices are the generalization of the connecting homomorphisms for an attractor–repeller pair to Morse decompositions. For our purposes it is sufficient to recall that given a Morse decomposition, a corresponding *connection matrix* is a linear map defined on the graded vector spaces made up of the sum of the Conley indices of Morse sets in a Morse decomposition, that is

$$\Delta : \bigoplus_{p\in\mathcal{P}} CH_*(M(p)) \to \bigoplus_{p\in\mathcal{P}} CH_*(M(p)).$$

We shall write this as a matrix

$$\Delta = [\Delta(p,q)]_{q,p\in\mathcal{P}}$$

where

$$\Delta(p,q) : CH_*(M(q)) \to CH_*(M(p)).$$

Furthermore, connection matrices satisfy the following conditions.

1. They are *upper triangular*, i.e. if $p \not> q$ then $\Delta(p,q) = 0$.

2. They are *boundary operators*, i.e. they are degree -1 maps

$$\Delta(p,q)CH_n(M(q)) \subset CH_{n-1}(M(p)),$$

   and they square to zero, $\Delta \circ \Delta = 0$.

3. If p and q are adjacent in the flow defined order then the connection matrix entry $\Delta(p,q)$ equals the connecting homomorphism for the attractor repeller pair $(M(p), M(q))$ of $M(q,p)$, i.e.

$$\Delta(p,q) = \partial(M(q), M(p)).$$

4. Recall that for any interval I in $\mathcal{P}$, the set $M(I)$ is an isolated invariant set, and hence, $CH_*(M(I))$ is defined. The relation between the indices of the Morse sets in $M(I)$, and $CH_*(M(I))$ is

$$CH_*(M(I)) \approx \frac{Ker\Delta(I)}{Image\Delta(I)}$$

   where

$$\Delta(I) = [\Delta(p,q)]_{q,p\in I}.$$

The following theorem, due to Franzosa [3], is fundamental.

Theorem 2.3 *Given a Morse decomposition, there exists at least one connection matrix.*

Returning now to the realm of differential equations, consider the 1-parameter family of equations

$$\dot{x} = F(x, \lambda) \qquad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$$

which we assume generates a continuous family of flows

$$\psi^\lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

Let $\Lambda = [-1-\mu, 1+\mu]$. The parameterized flow over Λ will be denoted by $\Psi : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \times \Lambda$ where

$$\Psi(t, x, \lambda) = (\psi^\lambda(t, x), \lambda).$$

We will make the following assumptions on the parameterized flow.

H1 *N is an isolating neighborhood for Ψ.*

H2 *A Morse decomposition of $K = \mathrm{Inv}(N, \Psi)$ is given by*

$$\mathcal{M}(K) = \{M(p) \mid p = 1, \ldots, P\}$$

with an admissible ordering

$$P > P - 1 > \ldots > 1.$$

Furthermore, for each $p = 1, \ldots, P$, the homotopy Conley index of each Morse set is

$$h(M(p)) \sim \Sigma^m \quad \text{for some} \quad m.$$

As before we shall adopt the notation that

$$N^\lambda = N \cap (\mathbb{R}^n \times \{\lambda\})$$

and

$$M^\lambda(p) = M(p) \cap (\mathbb{R}^n \times \{\lambda\}).$$

H3

$$K^{\pm 1} = \bigcup_{p=1}^{P} M^{\pm 1}(p).$$

Observe that **H3** implies that there are no connecting orbits between the Morse sets under the flows $\psi^{\pm 1}$.

Now consider a system of equations of the form

$$\begin{aligned} \dot{x} &= f(x, \lambda), \\ \dot{\lambda} &= \epsilon g(\lambda), \end{aligned} \tag{7}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $(1 - \lambda^2)g(\lambda) > 0$ for $\lambda \neq \pm 1$. Though, formally, equations (7) resemble those of (5) they are not the same since the Morse decompositions at $\lambda = \pm 1$ differ. In the application of Theorem 2.4 (stated below) to the proof of Theorem 1.2 the equations (7) are obtained by choosing a covering of the isolating neighborhood on which (5) are defined.

We shall denote the flow generated by (7) by

$$\Psi_\epsilon : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}.$$

Let $K_\epsilon = \mathrm{Inv}(N, \Psi_\epsilon)$. Observe that under Ψ_ϵ the Morse decomposition consists of $2P$ Morse sets

$$\mathcal{M}(K_\epsilon) = \{M(p^\pm) \mid p = 1, \ldots P\} \tag{8}$$

where $M(p^\pm) := M^{\pm 1}(p) \times \{\pm 1\} \subset \mathbb{R}^n \times \Lambda$. As was shown in [16], for $0 < |\epsilon| < \bar{\epsilon}$, the connection matrix for the Morse decomposition $\mathcal{M}(K_\varepsilon)$ has the form

$$\Delta_\varepsilon = \begin{bmatrix} 0 & T_\varepsilon \\ 0 & 0 \end{bmatrix}. \tag{9}$$

The submatrix T_ϵ will be called the *singular transition matrix* associated with Ψ_ε. Observe that:

$$\text{if } \epsilon < 0 \text{ then} \qquad T_\epsilon : \bigoplus_{p=1}^{P} CH_*(M(p^+)) \to \bigoplus_{p=1}^{P} CH_*(M(p^-));$$

$$\text{if } \epsilon > 0 \text{ then} \qquad T_\epsilon : \bigoplus_{p=1}^{P} CH_*(M(p^-)) \to \bigoplus_{p=1}^{P} CH_*(M(p^+)).$$

Hence, nonzero entries in the transition matrices correspond to $M(q^+) \to M(p^-)$ or $M(q^-) \to M(p^+)$ orbits for some p, q.

For the purpose of the following discussion let $\epsilon < 0$ (the case $\epsilon > 0$ is analogous). Let

$$T_\epsilon = [T_\epsilon(p,q)]_{q,p=1,\dots,P}$$

where

$$T_\epsilon(p,q) : CH_*(M(q^+)) \to CH_*(M(p^-)).$$

Since connection matrices are degree -1 operators

$$T_\epsilon(p,q) : CH_{n+1}(M(q^+)) \to CH_n(M(p^-)).$$

However, by [16, Theorem 5.4], or [11, Theorem 2.10]

$$CH_{n+1}(M(q^+)) \approx \tilde{H}_{n+1}(\Sigma h(M^1(q))) \approx CH_n(M^1(q)) \tag{10}$$

where $\Sigma h(M^1(q))$ denotes the suspension of the homotopy Conley index of $M^1(p)$ under the flow ψ^1 and

$$CH_n(M^{-1}(q)) \approx CH_n(M(q^-)).$$

We shall indicate these isomorphism by

$$\Sigma_n(q^+) : CH_n(M^1(q)) \to CH_{n+1}(M(q^+)).$$

and

$$\sigma_n(q^-) : CH_n(M^{-1}(q)) \to CH_n(M(q^-)).$$

Similarly if $\epsilon > 0$ then there exist isomorphisms

$$\Sigma_n(q^-) : CH_n(M^{-1}(q)) \to CH_{n+1}(M(q^-))$$

and

$$\sigma_n(q^+) : CH_n(M^1(q)) \to CH_n(M(q^+)).$$

Let

$$\Sigma_*^\pm := \bigoplus_{q=1}^P \Sigma_*(q^\pm) : \bigoplus_{q=1}^P CH_*(M^{\pm 1}(q)) \to \bigoplus_{q=1}^P CH_*(M(q^\pm))$$

and

$$\sigma_*^\pm := \bigoplus_{q=1}^P \sigma_*(q^\pm) : \bigoplus_{q=1}^P CH_*(M^{\pm 1}(q)) \to \bigoplus_{q=1}^P CH_*(M(q^\pm))$$

An important result which is proven in [9] is the following.

Theorem 2.4

$$(\sigma_*^-)^{-1} \circ T_{-\epsilon} \circ \Sigma_*^+ = (\Sigma_*^-)^{-1} \circ T_\epsilon^{-1} \circ \sigma_*^+.$$

Furthermore, as is, also, shown in [9] one can choose bases for all the spaces such that on the level of matrices this equality can be written as

$$T_{-\epsilon} = T_\epsilon^{-1}.$$

We shall from now on assume that this has been done.

3 Φ_ε for $\varepsilon < 0$.

This section contains the proof of Theorem 1.2(a), i.e., that for $\varepsilon < 0$

$$\#C(B_\varepsilon^+, B_\varepsilon^-; N, \Phi_\varepsilon) \geq 2$$

and addresses the question of the structure of these connection orbits. For $|\varepsilon| < \bar{\varepsilon}$, let $S_\varepsilon := \mathrm{Inv}(N, \Phi_\varepsilon)$.

Lemma 3.1 *For $\varepsilon < 0$,*

$$\mathcal{M}(S_\varepsilon) = \{B_\varepsilon^\pm, A^\pm\}$$

is a Morse decomposition of S_ε with an admissible ordering

$$A^+ > B_\varepsilon^+ > B_\varepsilon^- > A^-.$$

Proof. The planes $\lambda = \pm 1$ are invariant under Φ_ε and the flows on these planes are given by $\varphi^{\pm 1}$. Since $\varepsilon < 0$ and $(1-\lambda^2)g(\lambda) > 0$

$$(\mathrm{Inv}(N^{-1}, \varphi^{-1}), \mathrm{Inv}(N^1, \varphi^1))$$

forms an attractor-repeller pair decomposition of S_ε. Thus, by **A4**, $\{B_\varepsilon^\pm, A^\pm\}$ is a Morse decomposition of Φ_ε. The admissibility of the ordering is guaranteed by **A4**, $\varepsilon < 0$, and $(1-\lambda^2)g(\lambda) > 0$. □

Let

$$\begin{aligned}\Delta_\varepsilon : & CH_*(A^-) \oplus CH_*(B_\varepsilon^-) \oplus CH_*(B_\varepsilon^+) \oplus CH_*(A^+) \\ & \to CH_*(A^-) \oplus CH_*(B_\varepsilon^-) \oplus CH_*(B_\varepsilon^+) \oplus CH_*(A^+)\end{aligned}$$

denote a connection matrix for $\mathcal{M}(S_\varepsilon)$. Let $\Delta_\varepsilon(Y,X) : CH_*(X) \to CH_*(Y)$ denote the corresponding submatrix of Δ_ε.

Lemma 3.2 *For* $-\bar{\varepsilon} \leq \varepsilon < 0$

$$\Delta_\varepsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. Under the flow Φ_ε, the Conley indices for the equilibria are as follows:

$$\begin{aligned} CH_*(A^-) &= (\mathbb{Z}_2, 0, 0, \ldots) \\ CH_*(B_\varepsilon^-) &= (0, \mathbb{Z}_2, 0, \ldots) \\ CH_*(B_\varepsilon^+) &= (0, 0, \mathbb{Z}_2, 0, \ldots) \\ CH_*(A^+) &= (0, 0, 0, \mathbb{Z}_2, 0, \ldots). \end{aligned}$$

Since connection matrices are degree -1 operators, the only possible non-zero entries are $\Delta_\varepsilon(A^-, B_\varepsilon^-)$, $\Delta_\varepsilon(B_\varepsilon^-, B_\varepsilon^+)$ and $\Delta_\varepsilon(B_\varepsilon^+, A^+)$. By **A4** and [16], $\Delta_\varepsilon(A^-, B_\varepsilon^-) = 1$ and $\Delta_\varepsilon(B_\varepsilon^+, A^+) = 1$. Since $\Delta_\varepsilon \circ \Delta_\varepsilon = 0$, $\Delta_\varepsilon(B_\varepsilon^-, B_\varepsilon^+) = 0$. □

The proof of Theorem 1.2(a) will follow from the fact that $\Delta_\varepsilon(B_\varepsilon^-, B_\varepsilon^+) = 0$. Let

$$\Gamma_\varepsilon(j) := \{\gamma_\varepsilon \mid \theta(\gamma_\varepsilon) = j\}.$$

Lemma 3.3 *For* $\varepsilon \in [-\bar{\varepsilon}, 0)$, $B_\varepsilon^+ \cup B_\varepsilon^- \cup \Gamma_\varepsilon(j)$ *is an isolated invariant set.*

Proof. Suppose that for some $\varepsilon \in [-\bar{\varepsilon}, 0)$ and some $j \geq 0$, $B_\varepsilon^+ \cup B_\varepsilon^- \cup \Gamma_\varepsilon(j)$ is not compact in D.

Let $\{\gamma_n \mid n = 0, 1, 2, \ldots\} \subset \Gamma_\varepsilon(j)$ and let $\bar{\gamma}_n = B_\varepsilon^+ \cup B_\varepsilon^- \cup \gamma_n$. Since $\varepsilon \in [-\bar{\varepsilon}, 0)$, N is an isolating neighborhood and hence $\bar{\gamma}_n \subset N$. Since N is compact, there exists a subsequence $\{\bar{\gamma}_{n_i}\}$ such that $\bar{\gamma}_{n_i} \to \gamma$ as $i \to \infty$ where $\gamma \subset N$ but $\gamma \not\subset D$. The latter implies that $\gamma \cap \{A\} \times [-1-\mu, 1+\mu] \neq \emptyset$. However, since each $\bar{\gamma}_{n_i}$ is a compact connected invariant set, so is γ. Therefore $\gamma \cap \{A\} \times [-1-\mu, 1+\mu]$ consists of A^+, A^-, or $\{A\} \times [-1, 1]$. Assume $A^+ \in \gamma \cap \{A\} \times [-1-\mu, 1+\mu]$. Since $\varepsilon < 0$, $B^+ \in \gamma$, and γ is connected, there exists a connecting orbit from B^+ to A^+ which contradicts **A3**. A similar argument implies that $A^- \notin \{A\} \times [-1-\mu, 1+\mu]$. Therefore $\gamma \cap \{A\} \times [-1-\mu, 1+\mu] = \emptyset$ which is a contradiction. □

Let $D_\varepsilon(j) \subset D$ be an isolating neighborhood for Φ_ε such that

$$\mathrm{Inv}(D_\varepsilon(j)) = B_\varepsilon^+ \cup B_\varepsilon^- \cup \Gamma_\varepsilon(j).$$

and let $\partial(B_\epsilon^+, B_\epsilon^-; j)$ denote the connecting homomorphism for the index triple associated with the attractor-repeller pair $(B_\varepsilon^-, B_\varepsilon^+)$ decomposition of $\mathrm{Inv} D_\varepsilon(j)$.

Since we are using $\mathbb{Z}_2$ coefficients for the Conley index $\partial(B_\epsilon^+, B_\epsilon^-; j) = 0$ or 1. A standard index theory result is the following.

Lemma 3.4 $\partial(B_\epsilon^+, B_\epsilon^-; j) = 1 \Leftrightarrow CH_*(\mathrm{Inv} D_\varepsilon(j)) \approx 0 \Rightarrow \Gamma_\varepsilon(j) \neq \emptyset.$
$\partial(B_\epsilon^+, B_\epsilon^-; j) = 0 \Leftrightarrow CH_*(\mathrm{Inv} D_\varepsilon(j)) \approx CH_*(B_\varepsilon^-) \oplus CH_*(B_\varepsilon^+).$

Lemma 3.5 *For $\epsilon \in [-\bar{\epsilon}, 0)$,*

$$\partial(B_\epsilon^+, B_\epsilon^-; j) = \partial(B_{-\bar{\epsilon}}^+, B_{-\bar{\epsilon}}^-; j)$$

i.e., it is independent of ϵ.

Proof. By Lemma 3.4, $\partial(B_\varepsilon^+, B_\varepsilon^-; j)$ is determined by $CH_*(\mathrm{Inv} D_\varepsilon(j))$. Clearly, the Conley index of $\mathrm{Inv} D_\epsilon(j)$ can change only if for some $\varepsilon \in [-\bar{\varepsilon}, 0)$, $\mathrm{Inv} D_\varepsilon(j)$ is not isolated. Since N is an isolating neighborhood for Φ_ε, $\varepsilon \in [-\bar{\varepsilon}, 0)$, $\mathrm{Inv} D_\varepsilon(j) \subset \mathrm{int} N$. Since the orbits of $\Gamma_\varepsilon(j)$ and $\Gamma_\varepsilon(k)$ generate different homotopy classes in D, the only way $\mathrm{Inv} D_\varepsilon(j)$ could cease to be isolated would be for a connecting orbit to approach $A \times [-1-\mu, 1+\mu]$. However, as was demonstrated in Lemma 3.3, this is impossible for $\varepsilon < 0$. □

Lemma 3.6 *There exists a positive integer J such that if $|j| > J$, then $\partial(B_\varepsilon^+, B_\varepsilon^-; j) = 0$.*

Proof. By Lemma 3.4, if $\Gamma_\varepsilon(j) = \emptyset$ then $\partial(B_\varepsilon^+, B_\varepsilon^-; j) = 0$. By Lemma 3.5, it is sufficient to show that $\Gamma_{\bar{\varepsilon}}(j) = \emptyset$. $\Gamma_{\bar{\varepsilon}}(j)$ is bounded away from $A \times [-1, 1]$, hence if $\gamma_{\bar{\varepsilon}}$ denotes a $B_\varepsilon^+ \to B_\varepsilon^-$ connecting orbit, then there exists a maximal number of times that $\gamma_{\bar{\varepsilon}}$ can wind around the line $A \times [-1+\delta, 1-\delta]$ for a fixed $\delta > 0$. By choosing δ sufficiently small, we can approximate the orbit of $\gamma_{\bar{\varepsilon}}$ over the regions $-1 \leq \lambda \leq -1+\delta$ and $1-\delta \leq \lambda \leq 1$ via the flow φ^{-1} and φ^1 respectively. By **A4**, $\gamma_{\bar{\varepsilon}}$ cannot wind around A in $N^{\pm 1}$. Thus, the winding number of any $\gamma_{\bar{\varepsilon}}$ is bounded above. Therefore, for J sufficiently large, if $|j| > J$ then $\Gamma_{\bar{\varepsilon}}(j) = \emptyset$. □

Lemma 3.7 $\partial(B_\varepsilon^+, B_\varepsilon^-; 0) = 1$ *for any* $\varepsilon \in [-\bar{\varepsilon}, 0)$.

Proof. By Lemma 3.5 it is sufficient to show that $\partial(B_\varepsilon^+, B_\varepsilon^-; 0) = 1$ for some value of $\varepsilon \in [-\bar{\varepsilon}, 0)$. With this in mind, let Q_δ denote a δ-neighborhood of $\bigcup_{-1-\mu \le \lambda \le 1+\mu} B(\lambda, 0)$. By **A3** and [10], there exists $\varepsilon' > 0$ such that if $0 < |\varepsilon| < \varepsilon'$ then Q_δ is an isolating neighborhood for Φ_ε. Furthermore, $CH_*(\mathrm{Inv}(Q_\delta, \Phi_\varepsilon)) \approx 0$. Observe that if $\Gamma_\varepsilon(0) \subset Q_\delta$ then

$$CH_*(\mathrm{Inv}(Q_\delta, \Phi_\varepsilon)) = CH_*(\mathrm{Inv}\, D_\varepsilon(0))$$

and hence by Lemma 3.4, $\partial(B_\varepsilon^+, B_\varepsilon^-; 0) = 1$.

Thus, it remains to be shown that $\Gamma_\varepsilon(0) \subset Q_\delta$ for ε sufficiently small. By [16], in the Hausdorff metric

$$\lim_{\varepsilon \to 0} \Gamma_\varepsilon(0) = \Gamma$$

where Γ is a compact invariant set for Φ_0. Therefore, from **A5**, we have $\Gamma = \bar{B}(= \bigcup_{-1 \le \lambda \le 1} B(\lambda, 0))$, and hence $\Gamma_\varepsilon(0) \subset Q_\delta$ for ε sufficiently small. Since $\partial(B_\varepsilon^+, B_\varepsilon^-; 0) = 1$ is proven by the index computation as above, it remains unchanged as long as the isolation of $\Gamma_\varepsilon(0)$ is preserved. Thus, from Lemma 3.3, $\partial(B_\varepsilon^+, B_\varepsilon^-; 0) = 1$ must hold for all $\varepsilon \in [-\bar{\varepsilon}, 0)$. $\square$

Proof of Theorem 1.2(a). By Lemma 3.2 and Theorem 2.2, we have

$$0 = \Delta_\varepsilon(B_\varepsilon^-, B_\varepsilon^+) = \sum_{j \in \mathbb{Z}} \partial(B_\epsilon^+, B_\epsilon^-; j),$$

for any $\varepsilon \in [-\bar{\varepsilon}, 0)$.

The assumption **A5** implies that any $B_\varepsilon^+ \to B_\varepsilon^-$ connecting orbit with a negative winding number cannot exist for sufficiently small ε. Namely, there exists a sequence of negative real numbers $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$ such that for any $\varepsilon \in [\varepsilon_k, 0)$, no $B_\varepsilon^+ \to B_\varepsilon^-$ connecting orbit γ_ε with $-k \le \theta(\gamma_\varepsilon) < 0$ exists. Therefore for any $\varepsilon \in [\varepsilon_k, 0)$, we have

$$\sum_{j \in \mathbb{Z}} \partial(B_\epsilon^+, B_\epsilon^-; j) = \sum_{j \le -k-1} \partial(B_\epsilon^+, B_\epsilon^-; j) + 1 + \sum_{j \ge 1} \partial(B_\epsilon^+, B_\epsilon^-; j).$$

Here we have used Lemma 3.7 as well. From Lemma 3.5, $\partial(B_\epsilon^+, B_\epsilon^-; j)$ does not depend on ε for $\varepsilon \in [-\bar{\varepsilon}, 0)$, and hence, the above formula in fact holds for all $\varepsilon \in [-\bar{\varepsilon}, 0)$. Since $\bar{\varepsilon}$ does not depend on n, we conclude that

$$0 = 1 + \sum_{j=1}^{J} \partial(B_\epsilon^+, B_\epsilon^-; j). \tag{11}$$

Therefore, there exists at least one $j \in \{1, \ldots, J\}$ such that $\partial(B_\epsilon^+, B_\epsilon^-; j) = 1$. $\square$

Remark 3.8 Observe that there exist an odd number of positive winding numbers $j \geq 1$ for which $\partial(B_\epsilon^+, B_\epsilon^-; j) = 1$. These numbers will be denoted by

$$1 \leq j_1 < j_2 < \cdots < j_{2r+1}\,.$$

4 The Covering Flow

In the previous section the proof of Theorem 1.2(a) was reduced to equation (11) where the summation was indexed by elements of the fundamental group of D. Thus, to further exploit this equality, it is natural to consider covering spaces of the isolating neighborhoods. This is the focus of this section and will lead to the proof of Theorem 1.2(b).

Throughout this section, D will be considered to be the pointed space (D, B_ε^+). Let $\widetilde{D}$ denote the universal covering space of D with covering map

$$\rho : \widetilde{D} \to D\,.$$

There exists a unique local flow $\widetilde{\Phi}_\varepsilon$ defined on $\widetilde{D}$ which satisfies the relation

$$\rho(\widetilde{\Phi}_\varepsilon(t, v)) = \Phi_\varepsilon(t, \rho(v))\,. \tag{12}$$

Since $\pi_1(D) \approx \mathbb{Z}$, Φ_ε has infinitely many fixed points which will be denoted by $\{B_{\varepsilon,j}^\pm \mid j \in \mathbb{Z}\}$ where $\rho(B_{\epsilon,j}^\pm) = B_\epsilon^\pm$. $\widetilde{D}$ will be considered as the pointed space $(\widetilde{D}, B_{\varepsilon,0}^+)$. Furthermore, the notation is chosen such that if $B_{\varepsilon,j}^\pm$ is connected to $B_{\varepsilon,0}^\pm$ by a path α, then

$$[\rho(\alpha) \cup \bar{B}_\varepsilon] = j \in \mathbb{Z} \approx \pi_1(D).$$

Observe that this implies that if $\gamma_\varepsilon \in \Gamma_\varepsilon(j)$ then its lift $\widetilde{\gamma}_\varepsilon$ is a heteroclinic orbit $B^+_{\varepsilon,\ell} \to B^-_{\varepsilon,\ell+j}$.

By (12) one might expect that if K is an isolating neighborhood for Φ_ε then $\rho^{-1}(K)$ is an isolating neighborhood for $\widetilde{\Phi}_\varepsilon$. Unfortunately, $\rho^{-1}(K)$ need not be compact. This suggests the following definition. Fix sufficiently small $\mu > 0$. Let U be a μ-neighborhood of $\bar{B}(= \bar{B}_0,$ see Definition 1.1). Let $U \subset K \subset D$. Let $\widetilde{K}_k$ be a connected subset of $\widetilde{D}$ containing $\{B^\pm_{\varepsilon,j} \mid j = 0, \ldots k\}$ such that for every $x \in K \setminus U$, $\rho^{-1}(x) \cap \widetilde{K}_k$ consists of exactly k distinct elements and for every $x \in U$, $\rho^{-1}(x) \cap \widetilde{K}_k$ consists of exactly $k+1$ distinct elements. Observe that if $\alpha : [0,1] \to \widetilde{D}$ such that $\alpha(0) = B^+_{\varepsilon,0}$ and $\alpha(1) = B^-_{\varepsilon,0}$, then $0 \leq [\rho(\alpha) \cup \bar{B}_\varepsilon] \leq k$. It is now easily checked that if K is an isolating neighborhood for Φ_ε then $\widetilde{K}_k$ is an isolating neighborhood for $\widetilde{\Phi}_\varepsilon$.

From now on, let $K \subset D$ denote an isolating neighborhood for Φ_ε. Then for $\varepsilon \in [-\bar{\varepsilon}, 0)$,

$$\mathrm{Inv}(K, \Phi_\varepsilon) = B^+_\varepsilon \cup B^-_\varepsilon \cup C(B^+_\varepsilon, B^-_\varepsilon; N, \Phi_\varepsilon). \tag{13}$$

Observe that the arguments of Lemma 3.7 guarantee that such a K exists, i.e., that $C(B^+_\varepsilon, B^-_\varepsilon; N, \Phi_\varepsilon)$ is bounded away from $A \times [-1-\mu, 1+\mu]$ for all $\varepsilon \in [-\bar{\varepsilon}, 0)$. Let $\widetilde{S}_{\varepsilon,k} = \mathrm{Inv}(\widetilde{K}_k, \widetilde{\Phi}_\varepsilon)$.

Lemma 4.1 *For $\varepsilon < 0$,*

$$\mathcal{M}(\widetilde{S}_{\varepsilon,k}) = \{B^\pm_{\varepsilon,n} \mid 0 \leq n \leq k\}$$

is a Morse decomposition of $\widetilde{S}_{\varepsilon,k}$ with an admissible ordering

$$B^+_{\varepsilon,n} > B^-_{\varepsilon,m}, \quad m \geq n.$$

Proof. By equation (13) and the proof of Theorem 1.2(a), $\widetilde{S}_{\varepsilon,k}$ consists of the fixed points $\{B^\pm_{\varepsilon,n} \mid 0 \leq n \leq k\}$ and heteroclinic orbits from $B^+_{\varepsilon,n}$ to $B^-_{\varepsilon,m}$ where $m \geq n$. ∎

Let

$$\widetilde{\Delta}_{\varepsilon,k} : \bigoplus_{n=k}^{0} CH_*(B^-_{\varepsilon,n}) \oplus \bigoplus_{n=k}^{0} CH_*(B^+_{\varepsilon,n}) \to \bigoplus_{n=k}^{0} CH_*(B^-_{\varepsilon,n}) \oplus \bigoplus_{n=k}^{0} CH_*(B^+_{\varepsilon,n})$$

denote the connection matrix for $\mathcal{M}(\widetilde{S}_{\varepsilon,k})$.

Lemma 4.2

$$\widetilde{\Delta}_{\varepsilon,k} = \begin{bmatrix} 0 & T_{\varepsilon,k} \\ 0 & 0 \end{bmatrix}$$

where

$$T_{\varepsilon,k} : \bigoplus_{n=k}^{0} CH_*(B^+_{\varepsilon,n}) \to \bigoplus_{n=k}^{0} CH_*(B^-_{\varepsilon,n}).$$

Let $T_{\varepsilon,k}(m,n) : CH_*(B^+_{\varepsilon,n}) \to CH_*(B^-_{\varepsilon,m})$ *be the corresponding entry of* $T_{\varepsilon,k}$*. Then*

$$T_{\varepsilon,k}(m,n) = 1 \Leftrightarrow m - n = 0, j_1, j_2, \ldots, j_{2r-1}.$$

Proof. Connection matrices are strictly upper triangular with respect to the admissible ordering on the Morse decomposition. Hence, since $B^-_{\varepsilon,n} \not> B^-_{\varepsilon,m}$, $B^+_{\varepsilon,n} \not> B^+_{\varepsilon,m}$, and $B^+_{\varepsilon,n} > B^-_{\varepsilon,m}$, the only non-zero entries occur in $T_{\varepsilon,k}$. Let $\widetilde{D}_{\varepsilon,0}(j)$ denote the lift of $D_\varepsilon(j)$ such that B^+_ε is lifted to $B^+_{\varepsilon,0}$. Now observe that $CH_*(\widetilde{D}_{\varepsilon,0}(j)) \approx CH_*(D_\varepsilon(j))$. Therefore, $T_{\varepsilon,k}(j,0)$ can be identified with $\partial_\varepsilon(B^+_\varepsilon, B^-_\varepsilon, j)$. Hence, by Remark 3.8 $T_{\varepsilon,k}(j,0) = 1$ if and only if $j = 0, j_1, \ldots, j_{2r-1}$. The remaining non-zero entries are determined by the deck transformations of $\widetilde{D}_{\varepsilon,0}(j)$. □

Since $T_{\varepsilon,k}$ is independent of ε for $\varepsilon \in [-\bar{\varepsilon}, 0)$ from Lemma 4.2, we shall write

$$T_k = T_{\varepsilon,k}.$$

The same proof as in Lemma 4.1 gives:

Lemma 4.3 *For* $\varepsilon > 0$*,*

$$\mathcal{M}(\widetilde{S}_{\varepsilon,k}) = \{B^\pm_{\varepsilon,n} \mid 0 \le n \le k\}$$

is a Morse decomposition of $\widetilde{S}_{\varepsilon,k}$ *with an admissible ordering*

$$B^-_{\varepsilon,n} > B^+_{\varepsilon,m}, \quad m \ge n.$$

Proof of Theorem 1.2(b) Let $\gamma_{\varepsilon,\ell}$ denote a heteroclinic orbit $B^-_{\varepsilon,0} \to B^+_{\varepsilon,\ell}$ under $\widetilde{\Phi}_\varepsilon$. If $\ell \ne m$ then $\rho(\gamma_{\varepsilon,\ell}) \ne \rho(\gamma_{\varepsilon,m})$. Thus the theorem will be proven once it is shown that there are infinitely many values of ℓ for which a $B^-_{\varepsilon,0} \to B^+_{\varepsilon,\ell}$ orbit exists.

By Theorem 2.4 , for $\varepsilon > 0$

$$\widetilde{\Delta}_{\varepsilon,k} = \begin{bmatrix} 0 & T_k^{-1} \\ 0 & 0 \end{bmatrix} .$$

Let $R_k = T_k^{-1}$ and let $R_k(m,n) : CH_*(B^-_{\varepsilon,n}) \to CH_*(B^+_{\varepsilon,m})$ be the corresponding entry. Since $\widetilde{\Delta}_{\varepsilon,k}$ is a connection matrix, if $R_k(m,0) = 1$ then there exists a $B^-_{\varepsilon,0} \to B^+_{\varepsilon,m}$ orbit.

Recall from Lemma 4.2 that T_k is an upper triangular matrix whose diagonal consists of 1's, and whose $j_1, j_2, \ldots, j_{2r+1}$ super diagonals consist of 1's, and whose other entries are all zero. Thus, it is easy to check that if for some k_0, $R_{k_0}(m,0) = 1$, then for all $k \geq k_0$, $R_k(m,0) = 1$. Given k, let M_k denote the maximal integer such that $R_k(M_k, 0) = 1$. If $M_k \to \infty$ as $k \to \infty$, then the theorem is proven. So we can assume that there exists M such that $R_k(m,0) = 0$ for all $k, m > M$. However, this implies that for k sufficiently large

$$T_k(M + j_{2r+1}, M) \circ R_k(M, 0) = 1,$$

and hence $R_k \neq T_k^{-1}$, a contradiction.

5 Remarks

After submitting the first version of this paper, we became aware of the preprint by Doelman and Holmes [2]. The motivation of their work comes from study of travelling waves in the complex Ginzburg-Landau partial differential equations, and the results they obtained are similar to ours, although their approach is more geometric rather than topological, based on many drawings of the stable and unstable manifolds of the invariant manifold, combined with analytic information obtained from Melnikov-type computations. It is straightforward to verify, for the equation in [2], all the assumptions of this paper. Hence the results given here hold as well for it.

There is another interesting example to which one can apply the results in this paper. It is a model equation of "shallow water sloshing" studied by Hastings and McLeod [5], where the authors show the existence of various kinds of bounded solutions which exhibit a variety of types of oscillations. In a similar way for the model equation (2) and the equation in [2], it is easy to check the assumptions **A1**-**A5** for this equation, and hence it is possible to apply our results to the equation for "sloshing". Using our results, one may be able to show the existence of such periodic solutions which are characterized by certain symbolic sequences. The attempt to carry out this idea is in progress and will appear in a future publication [4].

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Letter to the first referee

The manuscript was revised in order to answer the inquiry of the first referee. The following is the list of corrections made in the revised version:

1. We included Lemma 3.3 and its proof which shows that the set $B_\varepsilon^+ \cup B_\varepsilon^- \cup \Gamma_\varepsilon(j)$ is an isolated invariant set.

2. The proof of Lemma 3.3 provides the key point of the proof of Lemma 3.5.

3. We have replaced the assumption **A5** with a stronger assumption which can be verified for systems with monotone winding around A, as was shown for the model equation (2) immediately after the new assumption **A5** is presented. Accordingly, the definition of the winding number is moved before the assumption **A5** is presented. Using the term "winding number", the statement of the main theorem is slightly strengthened.

4. Since we have changed the assumption **A5**, the remark which preceeds the proof of the statement (a) of the main theorem is no longer trivial. Therefore we have included the claim and its proof in the proof of the main theorem.

5. The definition of $\widetilde{K}_k$ is corrected.

6. Some minor typing errors are also corrected.

The authors are grateful for the careful reading of the referee, and hope that he/she is satisfied with these corrections.