AN INDEX CHANGING BIFURCATION CREATING HETERODIMENSIONAL CYCLES

SHIN KIRIKI, HIROSHI KOKUBU, AND MING-CHIA LI

Abstract. In this paper, we consider a three-dimensional diffeomorphism φ with a partially hyperbolic point p and a hyperbolic saddle point q, which has a codimension two bifurcation given by a period-doubling bifurcation at p, and at the same time, a quasi-transverse intersection between the strong stable manifold of p and the unstable manifold of q. We show that φ can be C^1 approximated by open sets $\mathcal{U}$ and $\mathcal{V}$ of diffeomorphisms ψ having two saddles converging to p and q, respectively, as $\psi \to \varphi$, such that (i) for any $\psi \in \mathcal{U}$, the two saddles are homoclinically related to each other; (ii) for any $\psi \in \mathcal{V}$, ψ has a robust heterodimensional cycle associated with two non-trivial hyperbolic sets containing the two saddles, respectively. The proof involves the detection of a local blender after the codimension two bifurcation.

1. Introduction

In [24], Palis conjectured that, for $r \geq 1$, any C^r diffeomorphism on a compact manifold is either hyperbolic or can be C^r approximated by diffeomorphisms exhibiting a homoclinic tangency or a heterodimensional cycle. The heterodimensional cycle means a heteroclinic cycle of hyperbolic periodic orbits with different unstable indices (dimensions of the unstable manifolds). For dimension higher than two, even the C^1 case of the Palis conjecture is still open (see [26, 27] for one and two dimensional cases). Inspired by the conjecture, we are interested in the dynamics and bifurcations that appear from a common boundary of hyperbolic dynamics and the class of diffeomorphisms with heterodimensional cycles. In this paper, we consider the coalescence of period-doubling bifurcations and heterodimensional cycles.

1.1. **A motivating example.** The destruction of two-dimensional horseshoe by local bifurcations and related dynamical phenomena were studied by many authors [14, 2, 8, 7, 16, 10]. In this paper, we begin by considering a deformation of a three-dimensional horseshoe in such a way that a period-doubling bifurcation occurs, say at p, which is one of the two fixed points of a horseshoe conjugate to the full shift of two symbols, while the other fixed point, denoted by q, remains hyperbolic. To be more concrete, consider a map Φ on $\mathbb{R}^3$ given by

$$(x, y, z) \mapsto (\lambda x, F(y, z)),$$

where λ is a real constant, and F is a diffeomorphism on $\mathbb{R}^2$ having the horseshoe as its maximal invariant set in $[-1, 1]^2$. Note that Φ is contracting and expanding uniformly along y and z-axes, respectively, and has the central direction along the x-axis. If the absolute value of λ is different from 1, Φ has a uniformly hyperbolic

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maximal Φ-invariant set Λ in the box $B = [-1, 1]^3$. Note that Λ coincides with a homoclinic class of Φ where the *homoclinic class* is the closure of all transverse intersections of the unstable and stable manifolds of every pair of periodic points for Φ. Especially, every periodic point in Λ is hyperbolic and has the identical index equal to one if $|\lambda| < 1$, or equal to two if $|\lambda| > 1$. Such a hyperbolic set Λ is called a three-dimensional horseshoe with index $k = 1, 2$, depending on the modulus of λ. Observe that $\Phi(B) \cap B$ consists of two vertical components $\Phi(H_1)$ and $\Phi(H_2)$, where H_1 and H_2 are disjoint horizontal subsets of B, which contain saddle fixed points p and q of index k, respectively (see Figure 1.1-(i) in the case of $k = 1$).

We now deform the diffeomorphism Φ in such a way that it creates two kinds of dynamical degeneracy, namely, a period-doubling bifurcation at p and a structurally unstable heteroclinic orbit between p and q. There are basically two cases of such a deformation:

(1) if the three-dimensional horseshoe has index one, namely, its unstable manifolds have dimension one, then we assume the period-doubling bifurcation at p to be supercritical (see in Definition 2.2). In this case, the other fixed point q remains to have index one. We also assume the existence of a quasi-transverse heteroclinic orbit from q to p (see Definition 2.6);
(2) if the three-dimensional horseshoe has index two, then the period-doubling bifurcation at p is assumed to be subcritical (see in Definition 2.2). In this case, we also assume the existence of a quasi-transverse heteroclinic orbit from p to q, rather than from q to p.

Notice that, in what follows, we may consider only the case (1) without loss of generality, because these two cases (1) and (2) can be interchanged with each other by inverting the diffeomorphism. We can realize the case (1) in a two-parameter family $\{\varphi_{\mu,\nu}\}_{\mu,\nu\in I}$ of diffeomorphisms in $\mathbb{R}^3$, where I is a small interval around 0, such that, for $\mu, \nu \in I$, if $(x, y) \in H_1$

$$\varphi_{\mu,\nu}(x, y, z) = (f_\mu(x), F(y, z)) = (-(1 + \mu)x + x^3 + \text{h.o.t.},\ F(y, z)),$$

and if $(x, y) \in H_2$

$$\varphi_{\mu,\nu}(x, y, z) = (f_\nu(x), F(y, z)) = (\nu + \lambda x + \text{h.o.t.},\ F(y, z)),$$

where h.o.t. stands for higher order terms and F is the same as that of Φ. Therefore, $\varphi_{\mu,\nu}$ has two fixed points $p_{\mu,\nu} = p_\mu \in H_1$ and $q_{\mu,\nu} = q_\nu \in H_2$ satisfying that:

- if μ increases and passes through 0, there occurs a period-doubling bifurcation at the fixed point p_μ that changes to a saddle fixed point of index two, and creates a period-two orbit of index one;
- q_ν remains a saddle fixed point of index one under variation of ν around 0.

Figures 1.1-(ii) illustrates the configuration of the fixed points and the periodic orbit after the bifurcation.

For $\mu, \nu = 0$, the diffeomorphism $\varphi_{\mu,\nu}$ has a heteroclinic cycle γ between the partially hyperbolic point p_0 and the hyperbolic saddle q_0, which is unfolded by the other parameter ν. The maximal invariant set of $\varphi_{0,0}$ in the box B, which we call a *period-doubling horseshoe*, is no longer uniformly hyperbolic due to the period-doubling bifurcation, while the maximal invariant set of $\varphi_{\mu,\nu}$ with $\mu < 0$ and $\nu \approx 0$ in B is topologically conjugate to the three-dimensional horseshoe with index one for Φ. In fact, the diffeomorphism $\varphi_{0,0}$ is located on the common boundary of the class of hyperbolic diffeomorphisms and that of diffeomorphisms with heterodimensional

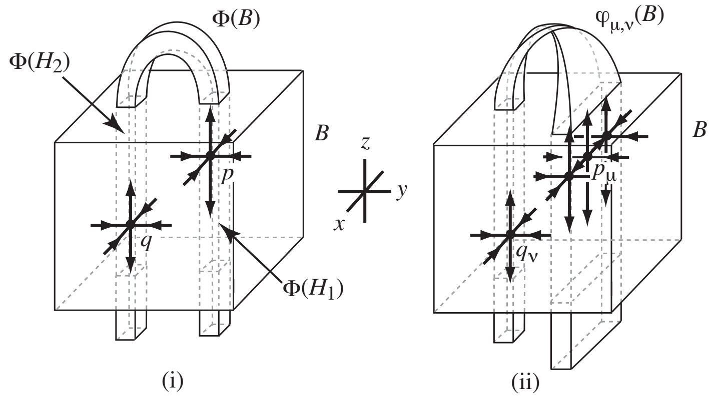

FIGURE 1.1

cycles, see the end of Subsection 2.2. Note that somewhat similar situations for saddle-node bifurcations were studied in [12, 15]. Moreover, although approaches are different from this paper, the fact that heterodimensional cycles yield saddle-node horseshoes was presented in [5, §4.1.1].

1.2. **Main results.** The above example motivates us to consider the following general situation: Let φ be a three-dimensional C^1 diffeomorphism having two fixed points p and q, such that q is a hyperbolic saddle fixed point with index one, while p is a partially hyperbolic fixed point having a central derivative equal to -1. Note that, after a suitable perturbation, p can undergo a supercritical period-doubling bifurcation, see Section 2. We also assume that these fixed points are contained in a heteroclinic cycle along which the one-dimensional strong unstable manifold $W^{uu}(p)$ and the two-dimensional stable manifold $W^s(q)$ intersect transversely, while the one-dimensional unstable manifold $W^u(q)$ and the one-dimensional strong stable manifold $W^{ss}(p)$ have a quasi-transverse intersection, namely, the tangent spaces of $W^u(q)$ and $W^{ss}(q)$ span a two-dimensional space at the intersection point, see Figure 1.2,

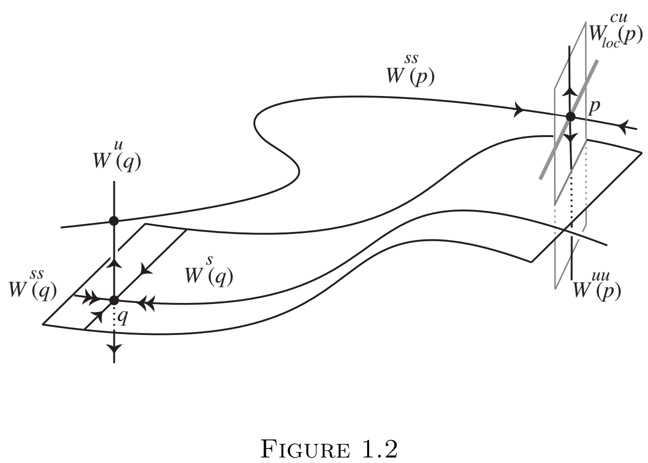

FIGURE 1.2

Let us state the main results of this paper. The precise definitions of several terms in the next theorems are all given in Section 2.

Theorem A. *Let φ be a three-dimensional C^1 diffeomorphism having a heteroclinic cycle between a partially hyperbolic fixed point p and a hyperbolic saddle fixed point q as above. Arbitrarily C^1-close to φ, there exists two disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ of diffeomorphisms such that (i) every diffeomorphism ψ in $\mathcal{U}$ has two saddle points p_ψ and q_ψ converging to p and q, respectively, as $\psi \to \varphi$, and these saddle points p_ψ and q_ψ are homoclinically related to each other; (ii) every diffeomorphism ψ in $\mathcal{V}$ has a robust heterodimensional cycle associated to two non-trivial hyperbolic sets containing two saddle points p_ψ and q_ψ converging to p and q, respectively, as $\psi \to \varphi$.*

Bonatti and Díaz proved in [5] that an unfolding of a heterodimensional cycle associated to two saddles generates a robust cycle associated to some non-trivial hyperbolic sets. However, the robust cycle proven in [5] may have nothing to do with the saddles in the initial cycle, see [5, Question 1.9]. One of the contributions of Theorem A is that, under a specific configuration considered in this paper, the new robust cycle is shown to be related to the initial saddles.

It is well known that a homoclinic tangency can be approximated by a diffeomorphism with transverse homoclinic intersections, and hence by a diffeomorphism with a horseshoe, due to the Birkhoff-Smale theorem, see [20, 22, 28]. Our second result asserts that an analogous situation can occur for a certain heterodimensional cycle, namely, it can be approximated by a certain invariant set called the blender introduced by Bonatti and Díaz [4]. See Subsection 3.2 for the definition of the blender. Note that similar results for a connected heterodimensional cycle were given essentially by Díaz and others [9, 4, 13, 6], however, a result for a non-connected heterodimensional cycle which is away from a connected one did not appear explicitly in their papers.

Theorem B. *Suppose φ be a three-dimensional C^1 diffeomorphism having a non-connected transverse heterodimensional cycle associated with saddle points p of index two and q of index one. Then, arbitrarily C^1-close to φ, there exists a diffeomorphism ψ having a local blender containing the continuation p_ψ. Moreover, there exists a robust heterodimensional cycle between the blender and a non-trivial hyperbolic set containing q_ψ.*

Note that Theorem B deals with dynamics generated from a codimension one cycle, while Theorem A does with that from a codimension two cycle, see Theorem 2.8 for a more precise formulation. However, Theorem B follows from a part of the argument in the proof of Theorem 2.8, because the part, namely the Assertion 4.1 and related Lemmas 4.3-4.4, does not depend on the particular situation of codimension two bifurcation.

This paper is organized as follows. In Section 2, after recalling some relevant definitions, we give a concrete statement of Theorem A, which is described by using the supercritical period-doubling bifurcation. In Section 3, we present a new version of the inclination lemma adapted to heterodimensional contexts. Also a geometric construction of the blender with its distinctive property and the C^1-creating lemma will be reviewed. Section 4 is entirely devoted to the proof of Theorem 2.8, but proofs of Lemmas 4.2-4.4 are given in Appendix A,

2. Definitions and settings

2.1. **Relevant definitions.** Let M be a smooth three-dimensional closed Riemannian manifold, and $\mathrm{Diff}^1(M)$ denote the set of C^1 diffeomorphisms on M.

2.1.1. *Partially hyperbolic sets.* We first note that most of discussions in this paper are supported by some kind of weak hyperbolicity which defined as follows:

Definition 2.1. For $\varphi \in \mathrm{Diff}^1(M)$, a φ-invariant set Λ is *strongly partially hyperbolic*, if there exists a $(d\varphi)_x$-invariant splitting $T_xM = \mathbb{E}^{ss}_x \oplus \mathbb{E}^c_x \oplus \mathbb{E}^{uu}_x$ for each $x \in \Lambda$, and constants $C > 0$, $0 < \lambda < 1$ satisfying the following conditions:

(1) The subspaces $\mathbb{E}^{ss}_x, \mathbb{E}^{uu}_x$ are non-trivial.
(2) For every $v_{ss} \in \mathbb{E}^{ss}_x$, $v_{uu} \in \mathbb{E}^{uu}_x$ and $n \geq 0$,
$$|(d\varphi^n)_x v_{ss}| \leq C\lambda^n |v_{ss}|, \quad |(d\varphi^{-n})_x v_{uu}| \leq C\lambda^n |v_{uu}|.$$
(3) There are $m > 0$ and $0 < K < 1$ such that for every unitary vector $v_{ss} \in \mathbb{E}^{ss}_x$, $v_{cu} \in \mathbb{E}^{cu}_x := \mathbb{E}^c_x \oplus \mathbb{E}^{uu}_x$,
$$|(d\varphi^m)_x v_{ss}||(d\varphi^{-m})_x v_{cu}| < K,$$
and every unitary vector $v_{cs} \in \mathbb{E}^{cs}_x := \mathbb{E}^{ss}_x \oplus \mathbb{E}^c_x$, $v_{uu} \in \mathbb{E}^{uu}_x$,
$$|(d\varphi^m)_x v_{cs}||(d\varphi^{-m})_x v_{uu}| < K,$$
In other words, $\mathbb{E}^{ss}_x \oplus \mathbb{E}^{cu}_x$ and $\mathbb{E}^{cs}_x \oplus \mathbb{E}^{uu}_x$ are both uniformly dominated (see [13, §2.2.1]).

In particular, if a periodic orbit is strongly partially hyperbolic in the above sense, it is called *partially hyperbolic*. A strongly partially hyperbolic set is (*uniformly*) *hyperbolic* if the center tangent space $\mathbb{E}^c_x$ is trivial for all $x \in \Lambda$; in this case, we have $T_xM = \mathbb{E}^s_x \oplus \mathbb{E}^u_x$ for each $x \in \Lambda$, where $\mathbb{E}^s_x = \mathbb{E}^{ss}_x$ and $\mathbb{E}^u_x = \mathbb{E}^{uu}_x$. We say that a φ-invariant set is *transitive*, if it is the closure of a single orbit. For any transitive hyperbolic set Λ (hence, dimensions of $\mathbb{E}^u_x$ for all $x \in \Lambda$ are identical), let $\mathrm{index}(\Lambda)$ denote the dimension of $\mathbb{E}^u_x$ for any $x \in \Lambda$.

2.1.2. *Period-doubling bifurcations.* In this subsection, we suppose φ is locally C^3 differentiable in a small neighborhood $U(p)$ of a fixed point p. Observe that, if $(d\varphi)_p$ has an eigenvalue of modulus one, from the above definition, $\dim \mathbb{E}^{ss}_p = \dim \mathbb{E}^c_p = \dim \mathbb{E}^{uu}_p = 1$.

Definition 2.2. We say that a partially hyperbolic fixed point p of φ undergoes a *supercritical period-doubling bifurcation* (see [28, §7.3]), if there exists a one-parameter family $\{\varphi_\mu\}$ of C^3 diffeomorphisms with $\varphi_0 = \varphi$ such that, for any μ near 0, φ_μ has a $(d\varphi_\mu)_{p_\mu}$-invariant splitting $T_{p_\mu}M = \mathbb{E}^c_{p_\mu} \oplus \mathbb{E}^{ss}_{p_\mu} \oplus \mathbb{E}^{uu}_{p_\mu}$ at a saddle fixed point p_μ with $\lim_{\mu\to 0} p_\mu = p_0 = p$ which satisfies the following conditions:

- the map φ_μ is given by a map $g_\mu(x) = g(x,\mu)$ in the $\mathbb{E}^c_{p_\mu}$-direction which satisfies $g(0,0) = 0$, $\frac{\partial g}{\partial x}(0,0) = -1$, $\frac{\partial^2 g}{\partial \mu \partial x}(0,0) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(0,0)\frac{\partial g}{\partial \mu}(0,0) < 0$, and $\frac{1}{3!}\frac{\partial^3 g}{\partial x^3}(0,0) + (\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(0,0))^2 > 0$. For simplicity, we may set $g_\mu(x) = -(1+\mu)x + x^3 + \text{h.o.t.}$ (see [21, II-4], [25, §3.2], [19, §4]);
- the map φ_μ is uniformly contracting in the $\mathbb{E}^{ss}_{p_\mu}$-direction, and uniformly expanding in the $\mathbb{E}^{uu}_{p_\mu}$-direction.

Similarly, we say that a partially hyperbolic fixed point p of φ undergoes the *subcritical* period-doubling bifurcation, if we assume $\frac{1}{3!}\frac{\partial^3 g}{\partial x^3}(0,0)+(\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(0,0))^2<0$ and the rest of the above assumptions are satisfied.

Note that, from the above definition, if a partially hyperbolic fixed point undergoes a supercritical period doubling bifurcation, the index of the fixed point changes from one for $\mu<0$ to two for $\mu>0$, and a pair of periodic points of period-two with index one appears for $\mu>0$. See Figure 2.1.

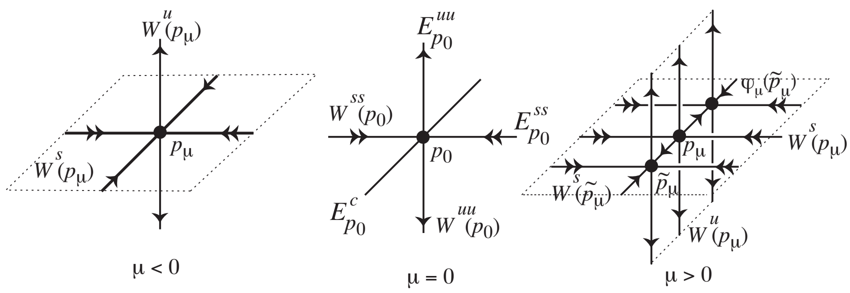

FIGURE 2.1

2.1.3. *Heteroclinic cycles.* For a C^1 diffeomorphism φ, let $W^u(z)$ and $W^s(z)$ be the unstable and stable manifolds of a hyperbolic fixed point z for φ, respectively. If z is partially hyperbolic, let $W^{ss}(z)$ and $W^{uu}(z)$ be the strong stable and strong unstable manifolds of z, which are tangent to $\mathbb{E}^{ss}_z$ and $\mathbb{E}^{uu}_z$, respectively. All of these are C^1 submanifolds and are invariant under φ.

Moreover, for a partially hyperbolic fixed point z, the generalized invariant manifold theorem asserts that there is a C^1 submanifold $W^{cu}(z)$ (resp. $W^{cs}(z)$) called the center unstable (resp. center stable) manifold of z, which is an invariant manifold tangent to $\mathbb{E}^{cu}_z:=\mathbb{E}^c_z\oplus\mathbb{E}^{uu}_z$ (resp. $\mathbb{E}^{cs}_z:=\mathbb{E}^c_z\oplus\mathbb{E}^{ss}_z$) at z. This is not unique but has a unique tangent space at z (see [30, Appendix III] or [25, p.158, Theorem 5]).

Definition 2.3. Let us define the followings.

- If φ has two hyperbolic periodic orbits $\mathcal{O}(p)$ of p and $\mathcal{O}(q)$ of q such that
$$W^u(\mathcal{O}(p))\cap W^s(\mathcal{O}(q))\neq\varnothing \text{ and } W^s(\mathcal{O}(p))\cap W^u(\mathcal{O}(q))\neq\varnothing,$$
then we say that φ has a *heteroclinic cycle* associated with these periodic orbits. In particular, if index$(\mathcal{O}(p))\neq$ index$(\mathcal{O}(q))$ holds, then the heteroclinic cycle is *heterodimensional.* Such a heteroclinic cycle is *connected* if $W^u(\mathcal{O}(p))\cap W^s(\mathcal{O}(q))$ or $W^u(\mathcal{O}(q))\cap W^s(\mathcal{O}(p))$ contains a φ-invariant curve that connects $\mathcal{O}(p)$ and $\mathcal{O}(q)$; otherwise, it is called *non-connected.*
- Similarly, in case that φ has a partially hyperbolic periodic orbit $\mathcal{O}(p)$ and a hyperbolic periodic orbit $\mathcal{O}(q)$, we say that φ has a *heteroclinic cycle* associated with these orbits, if one of the following conditions holds
  - $W^{uu}(\mathcal{O}(p))\cap W^s(\mathcal{O}(q))\neq\varnothing$ and $W^{ss}(\mathcal{O}(p))\cap W^u(\mathcal{O}(q))\neq\varnothing$;
  - $W^{ss}(\mathcal{O}(p))\cap W^u(\mathcal{O}(q))\neq\varnothing$ and $W^{uu}(\mathcal{O}(p))\cap W^s(\mathcal{O}(q))\neq\varnothing$.

Sometimes it is also referred to as a *bifurcating cycle*. Such a heteroclinic cycle is *connected* if there exists $W^{cu}(\mathcal{O}(p))$ (resp. $W^{cs}(\mathcal{O}(p))$) such that $W^{cu}(\mathcal{O}(p)) \cap W^s(\mathcal{O}(q))$ (resp. $W^{cs}(\mathcal{O}(p)) \cap W^u(\mathcal{O}(q))$) contains a φ-invariant curve that connects $\mathcal{O}(p)$ and $\mathcal{O}(q)$; otherwise, it is *non-connected*.

2.1.4. *Quasi-transverse intersection and crossing bifurcation.* In this paper, for given submanifolds S_1 and S_2 of M, the set of all transverse intersections between S_1 and S_2 is denoted by $S_1 \pitchfork S_2$, i.e., for all $x \in S_1 \pitchfork S_2$, $T_xS_1 + T_xS_2 = T_xM$. We say that S_1 and S_2 have a *quasi-transverse intersection*, if $S_1 \cap S_2 \neq \varnothing$ and there is an $x \in S_1 \cap S_2$ such that $\dim(T_xS_1 + T_xS_2) = \dim(M) - 1$.

Definition 2.4. Two (not necessarily distinct) hyperbolic saddle points p and q are said to be *homoclinically related*, if $W^u(p)\setminus\{p\} \pitchfork W^s(q)\setminus\{q\}$ and $W^u(q)\setminus\{q\} \pitchfork W^s(p)\setminus\{p\}$ are not empty. If $W^u(p) \pitchfork W^s(p) \neq \{p\}$, the closure of $W^u(p) \pitchfork W^s(p)$ is called the *homoclinic class* of p and is denoted by $H(p)$.

Note that, if p and q are homoclinically related, then $\mathrm{index}(p) = \mathrm{index}(q)$ and $H(p) = H(q)$ (see [6, 28]).

Let q be a hyperbolic fixed point of index one for φ with real eigenvalues. If the contracting eigenvalues of $(d\varphi)_q$ are distinct in modules, there is a strong stable foliation $\mathcal{F}^{ss}(q) \subset W^s(q)$ as in Figure 2.2 (see [25, p.158, Theorem 5] or [18]).

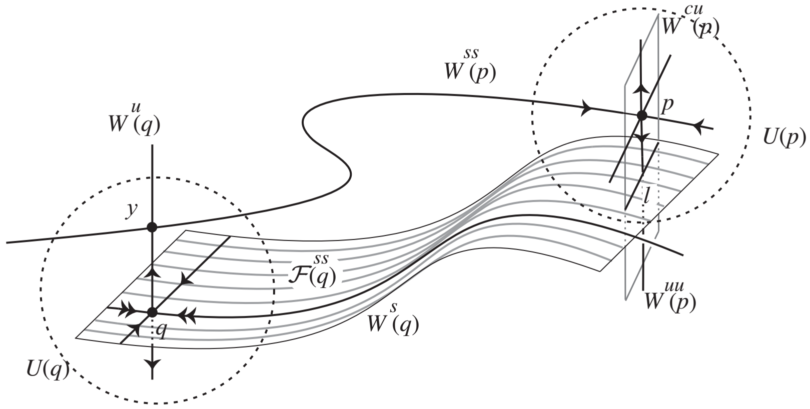

FIGURE 2.2

Definition 2.5. Consider a non-connected bifurcating cycle associated with a partially hyperbolic fixed point p and a hyperbolic fixed point q of index one with distinct contracting eigenvalues, as in Figure 2.2. The cycle is called *transverse*, if

(i) $W^{uu}(p) \setminus \{p\} \pitchfork W^s(q) \setminus W^{ss}(q) \neq \varnothing$;
(ii) there exists $W^{cu}(p)$ which contains a segment l of $W^s(q)$ such that, for any $x \in l$, the strong stable leaf through x intersects transversely with $W^{cu}(p)$;
(iii) $W^{ss}(p)$ is transverse a two-dimensional space tangent to $\mathbb{E}^{cu}_y$ at some $y \in W^{ss}(p) \cap W^u(q)$, where $\{\mathbb{E}^{cu}_y\}_{y \in W^u(q)}$ is a $d\varphi$-invariant continuous plane field along $W^u(q)$ with $\mathbb{E}^{cu}_y + \mathbb{E}^{ss}_y = T_yM$.

Definition 2.6. We say that a transverse bifurcating cycle for φ, associated with a partially hyperbolic fixed point p having a two-dimensional center unstable manifold and a hyperbolic fixed point q of index one, has a *crossing bifurcation* (see Figure 2.3), if there exists a one-parameter family $\{\varphi_\nu\}_{\nu\in I}$ of diffeomorphisms with $\varphi_0 = \varphi$, where I is an open interval containing 0, such that the following conditions are satisfied:

- $W^{ss}(p)$ and $W^u_{\mathrm{loc}}(q)$ have a quasi-transverse intersection point s_0 in $U(q)$, where $W^u_{\mathrm{loc}}(q)$ denotes the local unstable manifold of q. Moreover, there exist C^1 maps $s : I \to M$ with $s(0) = s_0$ and $t : I \to \mathbb{R}^+$ with $t(0) \neq 0$ such that $s(\nu) \in W^u_{\mathrm{loc}}(q_\nu)$, $\mathrm{dist}(s(\nu), W^{ss}(p_\nu)) = |\nu|t(\nu)$ for all $\nu \in I$, and $T_{s_0}M = T_{s_0}W^u_{\mathrm{loc}}(q_0) \oplus T_{s_0}W^{ss}(p_0) \oplus N$, where p_ν is the saddle point of index two converging to p as $\nu \to 0$, q_ν is the saddle continuation of q with $q_0 = q$ and N is the one-dimensional space spanned by nonzero $s'(0)$. See [13, §2.1.2] for more details.

Remark 2.7. Since the q_ν is hyperbolic saddle for any μ near 0, one has the $(d\varphi_\nu)_{q_\nu}$-invariant splitting $T_{q_\nu}M = \mathbb{E}^s_{q_\nu} \oplus \mathbb{E}^{ss}_{q_\nu} \oplus \mathbb{E}^{uu}_{q_\nu}$ at q_ν. Moreover, the existence of parameter-dependent smooth linearizing coordinates around q_ν are given in [29, Theorem 9 in §VIII] and [31, 32]. See also [5] for an affine representation along the so-called simple cycle which is arbitrarily C^1-close to a given heterodimensional cycle.

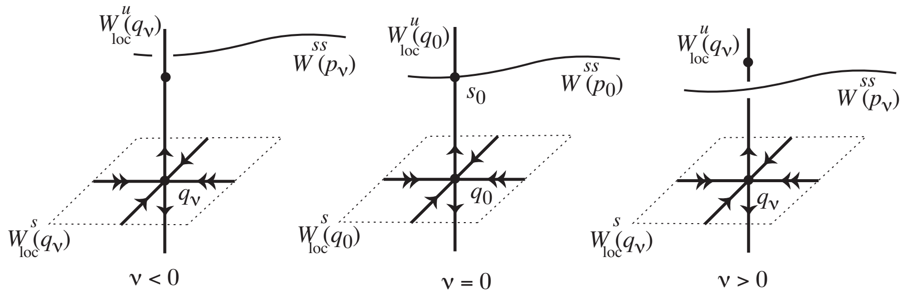

FIGURE 2.3

2.2. **Precise statements.** Keeping in mind the example of a breaking three-dimensional horseshoe in Section 1, consider a more general setting of diffeomorphism which has a bifurcating cycle between two fixed points p and q. When q is hyperbolic saddle, a saddle continuation of q is well-defined for any ψ close to φ, which is denoted by q_ψ. On the other hand, when p is partially hyperbolic, it can not be well-defined. Instead, we here consider a hyperbolic saddle point, which is also denoted by p_ψ, such that p_ψ converges p as $\psi \to \varphi$. Such a p_ψ is called a *related saddle* to p_φ. In what follows, the closure of a given set A is denote by $\mathrm{Cl}(A)$.

Theorem 2.8. *Suppose φ be a diffeomorphism on a three-dimensional manifold M which has a partially hyperbolic fixed point p undergoing the supercritical period-doubling bifurcation, and a hyperbolic fixed point q of index one. If φ has a transverse non-connected bifurcating cycle associated with p and q that has a crossing bifurcation, for every $n \in \mathbb{N}$, there exist disjoint open sets $\mathcal{U}$ and $\mathcal{V}_n \subset \mathrm{Diff}^1(M)$ with $\varphi \in \mathrm{Cl}(\mathcal{U}) \cap \mathrm{Cl}(\mathcal{V})$, where $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, satisfying the following statements:*

(1) *Every $\psi \in \mathcal{U}$ has related saddle p_ψ and continuation q_ψ, both of which are index one and homoclinically related to each other.*
(2) *There exists $\psi_n \in \mathrm{Diff}^1(M) \setminus \mathrm{Cl}(\mathcal{U})$, C^1-converges to φ as $n \to \infty$, having related saddles p_{ψ_n} of index two and $\tilde{p}_{\psi_n}$ with period-two of index one such that ψ_n^2 has a connected heterodimensional cycle associated with the saddles p_{ψ_n} and $\tilde{p}_{\psi_n}$.*
(3) *There exists a dense subset $\mathcal{D}_n \subset \mathcal{V}_n$ with $\psi_n \in \mathrm{Cl}(\mathcal{D}_n)$ such that every $\psi \in \mathcal{D}_n$ has a non-connected heterodimensional cycle associated with the related saddle p_ψ of index two and the saddle continuation q_ψ of index one. Moreover, the heterodimensional cycle is C^1-robust.*

Remark 2.9. To be precise, the C^1-robustness in the above (3) means as follows: there exist two hyperbolic sets Γ_ψ and Σ_ψ containing p_ψ and q_ψ, respectively, and a neighborhood $\mathcal{U}(\psi)$ of ψ such that every $\tilde{\psi} \in \mathcal{U}(\psi)$ has a heterodimensional cycle associated with continuations $\Gamma_{\tilde{\psi}}$ and $\Sigma_{\tilde{\psi}}$. See [5, Theorem 1.5, Question 1.9].

Remark 2.10. Analogous results to Theorem 2.8 should be obtained for non-generic bifurcations, e.g., the pitchfork bifurcation, on the center direction of p, instead of the period-doubling bifurcation.

The next assertion easily follows from Theorem 2.8 and [13, Theorem A].

Corollary 2.11. *For every $n \in \mathbb{N}$, let $\mathcal{V}_n$ be the open set given in (3) of Theorem 2.8. Then, every $\psi \in \mathcal{V}_n$ has a strongly partially hyperbolic transitive set Λ_ψ containing p_ψ, $\tilde{p}_\psi$ and q_ψ given in assertions (2) and (3). Moreover, the set Λ_ψ and homoclinic classes $H(p_\psi)$, $H(\tilde{p}_\psi)$ and $H(q_\psi)$ are all identical.*

Closing of this section, let us apply Theorem 2.8 to the example given in Section 1, which is given by the diffeomorphism φ having a period-doubling horseshoe including a heteroclinic cycle between partially hyperbolic fixed point p and hyperbolic fixed point q of index one. By Theorem 2.8, φ can be C^1-approximated by diffeomorphisms ψ and $\tilde{\psi}$ such that ψ has a three-dimensional horseshoe containing the related saddle p_ψ and the continuation q_ψ both of which are index one as in Figure 1.1-(i), whereas $\tilde{\psi}$ has a heterodimensional cycle associated to $p_{\tilde{\psi}}$ and $q_{\tilde{\psi}}$ as in Figure 1.1-(ii). Such a process of destructions of (non-normally hyperbolic) horseshoes could be more studied in general.

3. Auxiliary lemmas

3.1. **Heterodimensional inclination lemma.** We first present an extension of the inclination lemma. The original inclination lemma is as follows, which will be used in some proofs of the next section.

Lemma 3.1 (Inclination lemma)**.** *Let φ be a C^1 diffeomorphism on M having a saddle fixed point p. Let Σ be a C^1 submanifold of M with $\mathrm{index}(p) = \dim(\Sigma)$ such that Σ and $W^s(p)$ have a transverse intersection at $t \neq p$. Then, for any disk $D \subset W^u_{\mathrm{loc}}(p)$ containing p and for each $n > 0$, there exists a disk $\Delta^n \subset \varphi^n(\Sigma)$ such that Δ^n converges to D in the C^1-topology for $n \to \infty$.*

Proof. The original proof was given in [23, Lemma 1.1]. □

Note that the dimensions of $W^u(p)$ and Σ are supposed to be equal in the above lemma. It can be somewhat relaxed as in the next result in order to apply to heterodimensional situations of this paper.

Lemma 3.2 (heterodimensional inclination lemma)**.** *Let φ be a diffeomorphism on a three-dimensional manifold M having a saddle fixed point q with* $\mathrm{index}(q) = 1$ *such that φ is linearizable on a neighborhood of q. Assume that $\Sigma \subset M$ is a surface transverse to the strong stable foliation $\mathcal{F}^{ss} \subset W^s(q)$. Then, for any curve $L \subset W^u_{\mathrm{loc}}(q)$ containing q and for every $n > 0$, there exists a disk $\Sigma^n \subset \varphi^n(\Sigma)$ such that Σ^n converges to L in the C^1-sense as $n \to \infty$. Here the convergence in the C^1-sense means that, for any sequence $x_n \in \Sigma^n$ converging to a point $y \in L$ as $n \to \infty$, and for any covector $w_n \in T^*_{x_n}M$ vanishing on $T_{x_n}\Sigma^n$, one has that w_n, if appropriately normalized, converges to some covector in T^*_yM vanishing on $T_yW^{cu}_{\mathrm{loc}}(q)$ as $n \to \infty$, where the cotangent space $T^*_{x_n}M$ is defined to be the dual space of $T_{x_n}M$, and $W^{cu}_{\mathrm{loc}}(q)$ is the plane through q spanned by the unstable and weak stable directions.*

Proof. The eigenvalues of $(d\varphi)_q$ satisfy

$$0 < |\sigma_{ss}| < |\sigma_s| < 1 < |\sigma_u|.$$

Also, there exists a neighborhood $U(q)$ of q with C^1-coordinates (ξ, η, ζ) such that

$$(\varphi|U(q))(\xi, \eta, \zeta) = (\sigma_s\xi, \sigma_{ss}\eta, \sigma_u\zeta).$$

On the coordinates, each leaf of the local strong stable foliation $\mathcal{F}^{ss}_{\mathrm{loc}} \subset \mathcal{F}^{ss} \cap U(q)$ of q is a curve parallel to the η-axis, and $W^{cu}_{\mathrm{loc}}(q)$ is contained in the plane $\eta = 0$, as illustrated in Figure 3.1. We may suppose without loss of generality that Σ is compact and contained in $U(q)$.

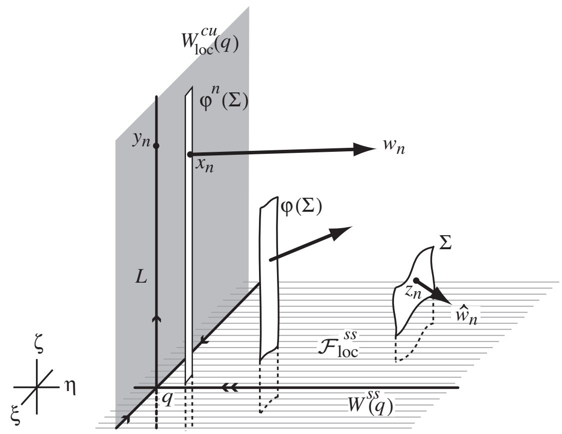

FIGURE 3.1

For a given $n > 0$, Σ^n is defined as a component of $\varphi^n(\Sigma) \cap U(q)$ which contains $\varphi^n(\Sigma \cap \mathcal{F}^{ss}_{\mathrm{loc}})$. It is straightforward to show that, if n is sufficiently large, $\varphi^n(\Sigma) \cap U(q)$ is arbitrarily close to $L \cap U(q)$ with respect to the Hausdorff metric. Hence, it remains to check the convergence from $\varphi^n(\Sigma) \cap U(q)$ to $L \cap U(q)$ in the C^1-sense.

Take any sequence $x_n \in \Sigma^n$ converging to some $y \in L \cap U(q)$ and $w_n \in T^*_{x_n}M$ with the kernel $\mathrm{Ker}(w_n) = T_{x_n}\Sigma$. Let $z_n = (\varphi^n)^{-1}(x_n)$ and $\hat{w}_n = (d\varphi^n)^*_{z_n}(w_n)$. where g^*_x denotes the pullback form $T^*_{g(x)}M$ to T^*_xM. Using the dual basis, write $w_n = {}^t(\xi_n, \eta_n, \zeta_n)$ and $\hat{w}_n = {}^t(\hat{\xi}_n, \hat{\eta}_n, \hat{\zeta}_n)$. From $\Sigma^n \subset \varphi^n(\Sigma) \cap U(q)$, it is clear that

$z_n \in \Sigma$ and $\hat{w}_n \in T^*_{z_n} M$. Moreover, we have $\mathrm{Ker}(\hat{w}_n) = T_{z_n}\Sigma$, since

$$\hat{w}_n(v) = (d\varphi^n)^*_{z_n}(w_n)(v) = w_n((d\varphi^n)_{z_n}(v)) = 0$$

for any $v \in T_{z_n}\Sigma$. Observe that the sequence $\{z_n\}$ accumulates to $\Sigma \cap W^s_{\mathrm{loc}}(q)$, and hence $T_{z_n}\Sigma$ is also transverse to the local strong stable foliation $\mathcal{F}^{ss}_{\mathrm{loc}}$. This means that there is a constant $C > 0$ such that

$$|\hat{\eta}_n| > C\ \min\{|\hat{\xi}_n|, |\hat{\zeta}_n|\}.$$

Now, noticing that the dual map $(d\varphi^n)^*_{z_n}$ can be expressed as the transpose ${}^t(d\varphi^n)_{z_n}$ in terms of the dual basis, we have

$$\begin{pmatrix} \xi_n \\ \eta_n \\ \zeta_n \end{pmatrix} = w_n = {}^t(d\varphi^n)^{-1}_{z_n}\hat{w}_n = \begin{pmatrix} \sigma_s^{-n}\hat{\xi}_n \\ \sigma_{ss}^{-n}\hat{\eta}_n \\ \sigma_u^{-n}\hat{\zeta}_n \end{pmatrix}.$$

Since

$$\frac{|\xi_n|}{|\eta_n|} = \frac{\sigma_{ss}^n|\hat{\xi}_n|}{\sigma_s^n|\hat{\eta}_n|} < \frac{1}{C}\left|\frac{\sigma_{ss}}{\sigma_s}\right|^n, \quad \frac{|\zeta_n|}{|\eta_n|} = \frac{\sigma_{ss}^n|\hat{\zeta}_n|}{\sigma_u^n|\hat{\eta}_n|} < \frac{1}{C}\left|\frac{1}{\sigma_u}\right|^n,$$

we have

$$\lim_{n\to\infty}\left|\frac{\xi_n}{\eta_n}\right| = 0, \quad \lim_{n\to\infty}\left|\frac{\zeta_n}{\eta_n}\right| = 0.$$

Thus, $w_n/|\eta_n|$ converges to $(0, 1, 0)$ as $n \to \infty$. □

3.2. **Blender structures.** We here review sufficient conditions which give blender structures in [4, §1] or [13, §3.5]. More general definitions of the blender are presented in [6, §6.2].

Consider a three-dimensional box $D = [-x_1, x_1] \times [-y_1, y_1] \times [-z_1, z_1] \subset \mathbb{R}^3$, as shown in Figure 3.2, where $x_1, y_1, z_1 > 0$ are constants. Let us denote each side face of D by

$$X^{\pm} = \{x = \pm x_1\} \cap D,\ Y^{\pm} = \{y = \pm y_1\} \cap D,\ Z^{\pm} = \{z = \pm z_1\} \cap D.$$

Let Π be either a curve or a surface in the interior of D. For any $p \in \Pi$ and a fixed $\delta \in (0, 1)$, define a *δ-cone* as

$$\mathcal{C}_\delta(p, \Pi) = \{v \in T_p\mathbb{R}^3 : v = v_1 + v_2, v_1 \in T_p\Pi, v_2 \in (T_p\Pi)^\perp, |v_2| \le \delta|v_1|\}.$$

Specially, we here denote by $\mathcal{C}^{uu}(p)$, $\mathcal{C}^u(p)$ and $\mathcal{C}^{ss}(p)$ the cone $\mathcal{C}_\delta(p, \Pi)$ if $T_p\Pi$ is along the z-axis, xz-plane and y-axis, respectively. See Figure 3.2.

We say that a segment L is *vertical* (resp. *horizontal*) *through* D if each endpoint of L is contained in either Z^+ or Z^- (resp. Y^+ or Y^-), and $T_xL \subset \mathcal{C}^{uu}(x)$ (resp. $\mathcal{C}^{ss}(x)$) for any $x \in L$. Let $\tilde{L}$ be a horizontal curve segments through D in the interior of D. Then there are two homotopy classes of vertical curve segments in $D \setminus \tilde{L}$. We say that a vertical curve segment L is *to the right* of $\tilde{L}$ if L dose not intersect $\tilde{L}$, and L is in the homotopy class of the curve segment $\{x_1\}\times\{0\}\times[-z_1, z_1]$ in $D \setminus \tilde{L}$.

We consider a diffeomorphism φ on $\mathbb{R}^3$ satisfying the following conditions (see [4, p. 365-366]):

(B1) There exists a connected component A of $D \cap \varphi(D)$ disjoint from $Y^{\pm}$ and $\varphi(X^{\pm} \cup Z^{\pm})$. Moreover, there exists an integer $n > 0$ and connected component B of $D \cap \varphi^n(D)$ such that B is disjoint from $Y^{\pm}$, X^+ and $\varphi^n(X^- \cup Z^{\pm})$. (See Figure 3.2 for the case of $n = 1$.)

(B2) There exist cone fields $\mathcal{C}^{ss}$, $\mathcal{C}^u$, $\mathcal{C}^{uu}$ and a constant $\rho > 1$ satisfying as follows:

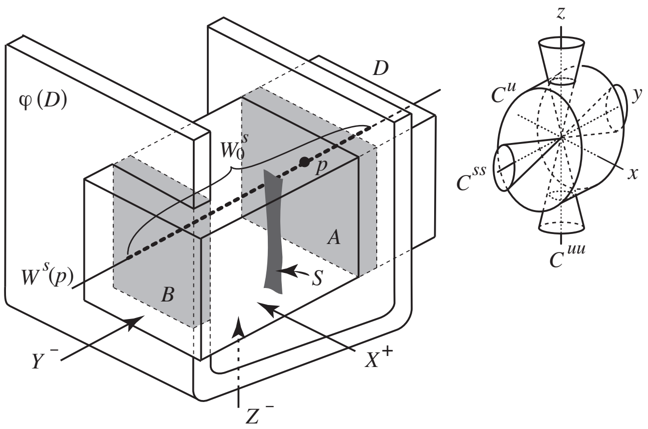

FIGURE 3.2

(B2.a) For every $x \in A$ (resp. $x \in B$) and every vector $v \in \mathcal{C}^{ss}(x)$, the vector $w = (d\varphi^{-1})_x v$ (resp. $w = (d\varphi^{-n})_x v$) belongs to the interior of $\mathcal{C}^{ss}(\varphi^{-1}(x))$ (resp. $\mathcal{C}^{ss}(\varphi^{-n}(x))$), and $|w| \geq \rho|v|$;

(B2.b) For every $x \in \varphi^{-1}(A)$ (resp. $x \in \varphi^{-n}(B)$) and every vector $v \in \mathcal{C}^u(x)$, the vector $w = (d\varphi)_x v$ (resp. $w = (d\varphi^n)_x v$) belongs to the interior of $\mathcal{C}^u(\varphi(x))$ (resp. $\mathcal{C}^u(\varphi^n(x))$), and $|w| \geq \rho|v|$;

(B2.c) For every $x \in \varphi^{-1}(A)$ (resp. $x \in \varphi^{-n}(B)$) the vector $w = (d\varphi)_x v$ (resp. $w = (d\varphi^n)_x v$) belongs to the interior of $\mathcal{C}^{uu}(\varphi^n(x))$, and also $|w| \geq \rho|v|$.

(B3) Let W_0^s be the connected component of $W^s(p) \cap D$ containing p with index$(p) = 2$, which is a unique saddle fixed point for φ^n in A. (The existence of such a unique fixed point is assured by the above (B1) and (B2), see [4, Lemma1.6].) There exists a neighborhood U^- of the side face X^- of D such that every vertical curve L through D at the right of W_0^s dose not intersect U^-.

(B4) There exist a neighborhood U^+ of the side face X^+ of D and a neighborhood V of W_0^s such that, for every vertical curve L through D to the right of W_0^s, one of the two following possibilities holds:

(B4.a) $\varphi(L) \cap A$ contains a vertical curve through D to the right of W_0^s and disjoint from U^+;

(B4.b) $\varphi^n(L) \cap B$ contains a vertical curve through D to the right of W_0^s and disjoint from V.

One can see that the maximal invariant set $\bigcap_{i \in \mathbb{Z}} \varphi^{in}(D)$ is uniformly hyperbolic, which is called a *blender* containing p. It has some significant feature about segments of $W^s(p) \cap D$ which are located in the right hand of W_0^s, as follows. We say that a two-dimensional disk $S \subset D$ is a *vertical strip* through D to the right of W_0^s, if the tangent space of S is contained in $\mathcal{C}^u$, and S is foliated by vertical segments through D to the right of W_0^s, whose tangent spaces are contained in $\mathcal{C}^{uu}$. A *width* $w(S)$ of S means the minimum length of curves in S transverse to $\mathcal{C}^{uu}$ joining the two components in the vertical boundary of S.

Lemma 3.3 (blender distinctive property)**.** *Let S be a vertical strip through D to the right of W_0^s. Then S meets transversely $W^s(p)$ in D even if $w(S)$ is arbitrary small.*

Proof. See [4, Lemma 1.8] or [6, Lemma 6.6]. □

3.3. **C^1-creating lemma for heterodimensional cycles.** In proofs of the next section, we will use a certain connecting lemma to show existence of heterodimensional cycles. Note that the original connecting lemma of [17] is stated about the creation of homoclinic points from almost homoclinic situations with respect to an isolated hyperbolic set. See also reformulations of the lemma in [33, 3].

Lemma 3.4. *Let p and q be saddle fixed points of $\varphi \in \mathrm{Diff}^1(\mathbb{R}^3)$ such that* $\mathrm{index}(p) = 2$ *and* $\mathrm{index}(q) = 1$*. Assume that $W^u(p,\varphi)$ and $W^s(q,\varphi)$ intersect transversely (note that in this case $W^u(p,\varphi) \cap W^s(q,\varphi)$ consists of C^1 curves), and $W^u(q,\varphi) \cap \mathrm{Cl}(W^s(p,\varphi)) \neq \varnothing$. Then for any C^1 neighborhood $\mathcal{U}$ of φ, there is a $\psi \in \mathcal{U}$ such that $\psi = \varphi$ on a neighborhood U of p and q, $W^u(p,\psi)$ and $W^s(q,\psi)$ intersect transversely, and $W^s(p,\psi) \cap W^u(q,\psi) \neq \varnothing$.*

Proof. The proof is obtained directly from [1, Lemma 2.8].[1)] □

4. Proof of the main theorems

4.1. **Existence of local blender structures.** Consider the two-parameter family $\{\varphi_{\mu,\nu}\}$ with $\varphi_{0,0} = \varphi$ of Theorem 2.8 which has a transverse non-connected heterodimensional cycle associated with $p_{0,0} = p$ and $q_{0,0} = q$. By some reparameterization, we may suppose that if ν is fixed near 0, the one-parameter subfamily $\{\varphi_\mu\}$ of $\{\varphi_{\mu,\nu}\}$ has a supercritical period-doubling bifurcation of $p_{\mu,\nu}$ for $\mu = 0$, while if μ is fixed near 0, the one-parameter subfamily $\{\varphi_\nu\}$ of $\{\varphi_{\mu,\nu}\}$ has a crossing bifurcation at s for $\nu = 0$. From the transverse condition of cycle (see Definition 2.5), there exist a small segment $W \subset W^{uu}(p)$ having a transverse intersection with $W^s(q) \setminus W^{ss}(q)$ and a small segment $\tilde{W} \subset W^{ss}(p)$ having a quasi-transverse intersection with $W^u(q)$ such that $\varphi^n(W)$ and $\tilde{W}$ have no intersection in a neighborhood $U(q)$ of q for any $n \geq 0$, see Figure 4.1. For such a family, we give the

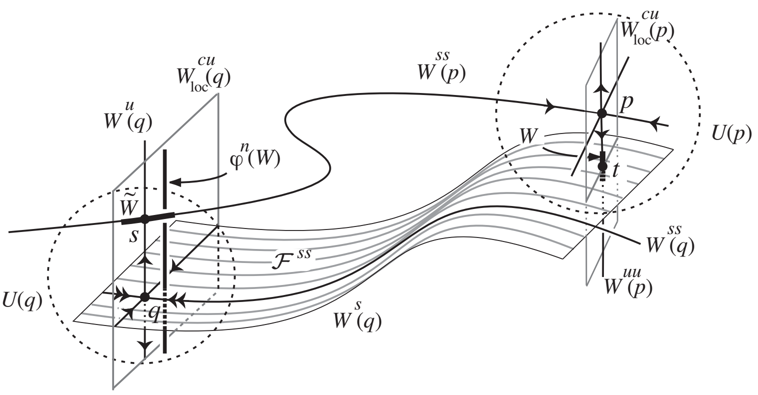

FIGURE 4.1

[1)] The authors are indebted to Prof. Lan Wen and anonymous referee for their comments.

following result.

Assertion 4.1. *There exist an open subset $\mathcal{O} \subset \mathbb{R}^2$ with $\mathrm{Cl}(\mathcal{O}) \ni (0,0)$, a box D in a neighborhood of $p_{\mu,\nu}$, and an integer $N > 0$ such that, for any $(\mu,\nu) \in \mathcal{O}$, $\varphi_{\mu,\nu}$ has a blender $\bigcap_{i\in\mathbb{Z}} \varphi_{\mu,\nu}^{iN}(D)$ containing $p_{\mu,\nu}$. Moreover, there exists a segment in $W^u(q_{\mu,\nu})$ which is vertical through D to the right of the segment of $W^{ss}_{\mathrm{loc}}(p_{\mu,\nu})$.*

The proof of Assertion 4.1 will be supported by the next Lemmas 4.2-4.4, where the crucial point is how to choose the return time N of the assertion. Note that, from this assertion, one can immediately show Theorem B. In fact, these lemmas still hold even if the saddle point $p_{\mu,\nu}$ with $\mathrm{index}(p_{\mu,\nu}) = 2$ is far from the local bifurcating situation.

When $\mu > 0$ and ν is near 0, by the undergoing of supercritical period-doubling bifurcation, there exists the hyperbolic splitting $T_{p_{\mu,\nu}}M = \mathbb{E}^c_{p_{\mu,\nu}} \oplus \mathbb{E}^{ss}_{p_{\mu,\nu}} \oplus \mathbb{E}^{uu}_{p_{\mu,\nu}}$ at $p_{\mu,\nu}$ such that $\varphi_{\mu,\nu}$ is expanding weakly in the direction of $\mathbb{E}^c_{p_{\mu,\nu}}$, contracting in the direction of $\mathbb{E}^{ss}_{p_{\mu,\nu}}$ and expanding strongly in the direction of $\mathbb{E}^{uu}_{p_{\mu,\nu}}$. The splitting is extended to $U(p_{\mu,\nu})$ which in the local chart is of the form

$$\mathbb{E}^c_{p_{\mu,\nu}} = \mathbb{R}^c \times \{(0^{ss}, 0^{uu})\}, \quad \mathbb{E}^{ss}_{p_{\mu,\nu}} = \{0^c\} \times \mathbb{R}^{ss} \times \{0^{uu}\} \quad \mathbb{E}^{uu}_{p_{\mu,\nu}} = \{(0^c, 0^{ss})\} \times \mathbb{R}^{uu}.$$

Consider a small box $D = D(\delta) = [-\delta,\delta]^3$ in $U(p_{\mu,\nu})$ for a small $\delta > 0$. Let $\Delta_0 \subset W^u(p_{\mu,\nu})$ be the section containing $p_{\mu,\nu}$, which is along the unstable plane $\{x^c = 0\}$, see Figure 4.2, and $t_{\mu,\nu}$ be a transverse point in $W^{uu}_{\mathrm{loc}}(p_{\mu,\nu}) \cap W^s(q_{\mu,\nu})$. We may assume that $t_{\mu,\nu}$ is contained in $U(p_{\mu,\nu})$. Define

$$k_0 := \min\left\{k \geq 0 \mid \varphi_{\mu,\nu}^{-k}(t_{\mu,\nu}) \in D\right\}.$$

When μ and ν are close to 0, one get the curve segment $l_{\mu,\nu}$ in $W^s(q_{\mu,\nu}) \cap \varphi_{\mu,\nu}^{k_0}(\Delta_0)$ containing $t_{\mu,\nu}$. Denote by $U(l_{\mu,\nu})$ a small neighborhood of $l_{\mu,\nu}$ as in Figure 4.2. Consider a two-dimensional connected subset $\Sigma_{\mu,\nu}$ of $W^u(p_{\mu,\nu}) \cap U(l_{\mu,\nu})$ containing $l_{\mu,\nu}$.

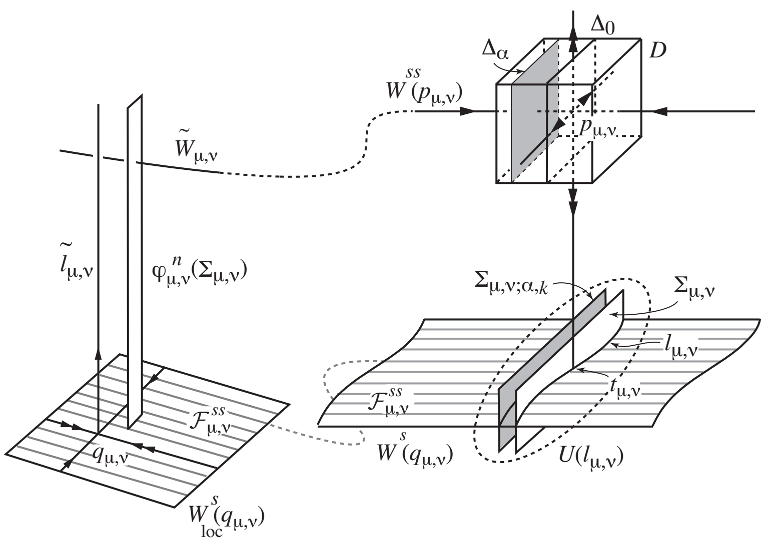

FIGURE 4.2

We here set $\nu = 0$. Let $\tilde{l}_{\mu,0}$ be a curve segment in $W^u_{\mathrm{loc}}(q_{\mu,0})$ such that $q_{\mu,0}$ and $s_{\mu,0}$ are contained in $\tilde{l}_{\mu,0}$, where $s_{\mu,0}$ is a quasi-transverse intersection between $W^{ss}(p_{\mu,0})$ and $W^u_{\mathrm{loc}}(q_{\mu,0})$. Also, let $\tilde{W}_{\mu,0}$ be a small curve segment in $W^{ss}(p_{\mu,0})$ such that $s_{\mu,0} \in \tilde{W}_{\mu,0}$. Since $\Sigma_{\mu,0} \pitchfork \mathcal{F}^{ss}_{\mu,0}$, for any large $n > 0$, $\varphi^n_{\mu,0}(\Sigma_{\mu,0})$ contains a strip $\Sigma^n_{\mu,0}$ which is C^1-close to $\tilde{l}_{\mu,0}$ in the sense of Lemma 3.2. We may suppose $\Sigma^n_{\mu,0} \cap \tilde{W}_{\mu,0} = \varnothing$ by choosing $\delta > 0$ smaller, if necessary. Therefore, by the condition of the crossing bifurcation and the intermediate value theorem, we get a large $n_1 > 0$ satisfying the following: for any $n \geq n_1$, there exists a parameter interval $J_{\mu,n}$ near 0 such that $\varphi^n_{\mu,\nu}(\Sigma_{\mu,\nu})$ and $\tilde{W}_{\mu,\nu}$ has a transverse intersection for any $\nu \in J_{\mu,n}$, see Figure 4.2, where $\tilde{W}_{\mu,\nu}$ is the continuation of $\tilde{W}_{\mu,0}$.

For any $|\alpha| \leq \delta$, consider a section Δ_α of D which is along the plane $\{x^c = \alpha\}$ as in Figure 4.2. For any $k \geq k_0$, we have a section $\Sigma_{\mu,\nu;\alpha,k}$ in $\varphi^k_{\mu,\nu}(\Delta_\alpha) \cap U(l_{\mu,\nu})$ which is C^1-close to $\Sigma_{\mu,\nu}$. Hence, for any $n \geq n_1$ and $\nu \in J_{\mu,n}$, $\varphi^n_{\mu,\nu}(\Sigma_{\mu,\nu;\alpha,k})$ and $\tilde{W}_{\mu,\nu}$ also has a transverse intersection which is denoted by $\tilde{t}_{\mu,\nu}$, see Figure 4.3.

For simplicity of description, let us introduce a notion of *angle* as follows. For given nonzero vector u and subspace V of TM, one can write $u = u_1 + u_2$ where $u_1 \in V$ and $u_2 \in V^\perp$, where $V^\perp$ is the orthogonal complement of V in TM. Hence, the angle between u and V can be defined as

$$\angle(u, V) := \begin{cases} |u_2|/|u_1|, & \text{if } |u_1| \neq 0, \\ \infty, & \text{if } |u_1| = 0. \end{cases}$$

The next lemma ensures existence of strong stable cone in (B2.a) of Section 3.2.

Lemma 4.2 (existence of $\mathcal{C}^{ss}$). *For any $\mu > 0$ and $n \geq n_1$, there exist integers $m_1 > 0$ and $k_1 \geq k_0$ satisfying the next assertion: if $k \geq k_1$, $m \geq m_1$, and $N^{ss} := k + m + n$, then for any $\nu \in J_{\mu,n}$ there exist components $A \subset D \cap \varphi_{\mu,\nu}(D)$ and $B \subset D \cap \varphi^{N^{ss}}_{\mu,\nu}(D)$, and exists a cone field $\mathcal{C}^{ss}$ on D such that, for any $x \in A$, resp. $x \in B$ and nonzero $u \in \mathcal{C}^{ss}(x)$,*

$$(d\varphi^{-1}_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^{ss}(\varphi^{-1}_{\mu,\nu}(x))), \quad |(d\varphi^{-1}_{\mu,\nu})_x u| \geq \rho_1 |u|, \tag{4.1}$$

$$\textit{resp.} \quad (d\varphi^{-N^{ss}}_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^{ss}(\varphi^{-N^{ss}}_{\mu,\nu}(x))), \quad |(d\varphi^{-N^{ss}}_{\mu,\nu})_x u| \geq \rho_1 |u| \tag{4.2}$$

where $\rho_1 > 1$ is some constant.

Since the proof of this lemma is straightforward and technical, it is postponed to Appendix A.

Next, we present Lemma 4.3 about the unstable cone field corresponding to that of (B2.b) in Section 3.2. For any $\mu > 0$, ν near 0, $k \geq k_1$ and $|\alpha| \leq \delta$, the intersection of $\Sigma_{\mu,\nu;\alpha,k}$ and the component of $W^s(q_{\mu,\nu}) \cap U(l_{\mu,\nu})$ containing $l_{\mu,\nu}$ is denoted by $l_{\mu,\nu;\alpha,k}$, see Figure 4.3. Let

$$\theta_1 := \min\left\{\angle(v, T_y\mathcal{F}^{ss}_{\mu,\nu}(y)) \;:\; y \in l_{\mu,\nu;\alpha,k},\ v \in T_y\Sigma_{\mu,\nu;\alpha,k}\right\}.$$

The next two auxiliary cones, as shown in Figure 4.3, will be applied to show the existence of unstable cone field $\mathcal{C}^u$ on D in the next lemma. For given $\mu > 0$, ν near 0, $k \geq k_1$ and $y \in U(l_{\mu,\nu}) \cap \varphi^k_{\mu,\nu}(D)$, there exists $|\alpha| \leq \delta$ such that $y \in \Sigma_{\mu,\nu;\alpha,k}$. For such a $y \in \Sigma_{\mu,\nu;\alpha,k}$ and any $z \in U(\tilde{l}_{\mu,\nu})$, define

$$\begin{aligned} \mathcal{C}_1(y) &= \{v \in T_yM \;:\; \angle(v, T_y\Sigma_{\mu,\nu;\alpha,k}) \leq \theta_1/2\}; \\ \mathcal{C}_2(z) &= \{w \in T_zM \;:\; \angle(w, (\mathbb{E}^{ss}_{q_{\mu,\nu}})^\perp) \leq 1/2\}. \end{aligned}$$

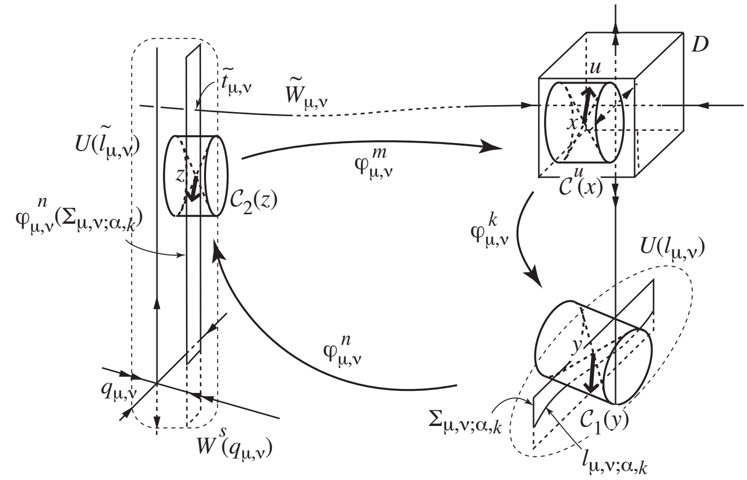

FIGURE 4.3

We here set $\nu = 0$ to get a constant n_2 satisfying as follows. Note that the one-dimensional space $\mathbb{E}^u_{q_{\mu,\nu}}$ dominates $\mathbb{E}^s_{q_{\mu,\nu}}$ for every ν near zero. Therefore, there exists an integer $n_2 \geq n_1$ such that for any $n \geq n_2$, $y \in \varphi^{-n}_{\mu,0}(U(\tilde{l}_{\mu,0})) \cap \Sigma_{\mu,0;\alpha,k}$ and $v \in \mathcal{C}_1(y)$,

$$(d\varphi^n_{\mu,0})_y v \in \mathcal{C}_2(\varphi^n_{\mu,0}(y)).$$

Lemma 4.3 (existence of $\mathcal{C}^u$)**.** *For any $\mu > 0$ and $n \geq n_2$, there exist integers $m_2 > m_1$ and $k_2 \geq k_1$ satisfying the next assertion: if $k \geq k_2$, $m \geq m_2$, and $N^u := k + m + n$, then for any $\nu \in J_{\mu,n}$ there exist components $A \subset D \cap \varphi_{\mu,\nu}(D)$, $B \subset D \cap \varphi^{N^u}_{\mu,\nu}(D)$, and cone field $\mathcal{C}^u$ on D such that, for any $x \in \varphi^{-1}_{\mu,\nu}(A)$, resp. $x \in \varphi^{-N^u}_{\mu,\nu}(B)$ and nonzero $u \in \mathcal{C}^u(x)$,*

$$(d\varphi_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^u(\varphi_{\mu,\nu}(x))), \quad |(d\varphi_{\mu,\nu})_x u| \geq \rho_2 |u|, \tag{4.3}$$

$$\textit{resp.} \quad (d\varphi^{N^u}_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^u(\varphi^{N^u}_{\mu,\nu}(x))), \quad |(d\varphi^{N^u}_{\mu,\nu})_x u| \geq \rho_2 |u| \tag{4.4}$$

where $\rho_2 > 1$ is some constant.

The proof is given in Appendix A.

The existence of the strong unstable cone field $\mathcal{C}^{uu}$ for the blender structure is shown below. Same setting as in Lemma 4.3, we also define the next auxiliary cone fields. For given $\mu > 0$, ν near 0, $k \geq k_2$ and $y \in U(l_{\mu,\nu}) \cap \varphi^k_{\mu,\nu}(D)$, there exists $|\alpha| \leq \delta$ such that $y \in \Sigma_{\mu,\nu;\alpha,k}$. For such a $y \in \Sigma_{\mu,\nu;\alpha,k}$, define

$$\tilde{\mathcal{C}}_1(y) = \{v \in T_y M \ : \ \angle((d\varphi^{-k}_{\mu,\nu})_y v, \mathbb{E}^{uu}_{p_{\mu,\nu}}) \leq \theta^u\},$$

where θ^u is the same value in the proof of Lemma 4.3. Then,

$$(d\varphi^{-k}_{\mu,\nu})_y \tilde{\mathcal{C}}_1(y) \subset \mathcal{C}^u(\varphi^{-k}_{\mu,\nu}(y)).$$

On the other hand, for any $z \in U(\tilde{l}_{\mu,\nu})$, define

$$\tilde{\mathcal{C}}_2(z) = \{w \in T_z M : \angle(w, \mathbb{E}^u_{q_{\mu,\nu}}) \leq 1/2\}.$$

Note that $\tilde{\mathcal{C}}_1(y) \subset \mathcal{C}_1(y)$ and $\tilde{\mathcal{C}}_2(z) \subset \mathcal{C}_2(z)$.

From the condition around $q_{\mu,\nu}$, $\mathbb{E}^u_{q_{\mu,\nu}}$ dominates $\mathbb{E}^{ss}_{q_{\mu,\nu}} \oplus \mathbb{E}^s_{q_{\mu,\nu}}$, while $\mathbb{E}^s_{q_{\mu,\nu}}$ dominates $\mathbb{E}^{ss}_{q_{\mu,\nu}}$ for every ν near zero. Hence, for $\nu = 0$, we get an integer $n_3 \geq n_2$ such that for any $n \geq n_3$, $y \in \varphi^{-n}_{\mu,0}(U(\tilde{l}_{\mu,0})) \cap \Sigma_{\mu,0;\alpha,k}$ and $v \in \tilde{\mathcal{C}}_1(y)$,

$$(d\varphi^n_{\mu,0})_y v \in \tilde{\mathcal{C}}_2(\varphi^n_{\mu,0}(y)).$$

Lemma 4.4 (existence of $\mathcal{C}^{uu}$). *For any $\mu > 0$ and $n \geq n_3$, let $m_3 := m_2(\mu, n)$ and $k_3 := k_2(\mu, n)$ where m_2 and k_2 are given in Lemma 4.2. If $k \geq k_3$, $m \geq m_3$, and $N^{uu} := k+m+n$, then, for any $\nu \in J_{\mu,n}$, there exist components $A \subset D \cap \varphi_{\mu,\nu}(D)$, $B \subset D \cap \varphi^{N_3}_{\mu,\nu}(D)$, and cone field $\mathcal{C}^{uu}$ on D such that, for any $x \in \varphi^{-1}_{\mu,\nu}(A)$, resp. $x \in \varphi^{-N^{uu}}_{\mu,\nu}(B)$ and nonzero $u \in \mathcal{C}^{uu}(x)$,*

$$(d\varphi_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^{uu}(\varphi_{\mu,\nu}(x))), \quad |(d\varphi_{\mu,\nu})_x u| \geq \rho_3 |u|,$$

$$\textit{resp.} \quad (d\varphi^{N^{uu}}_{\mu,\nu})_x u \in \mathrm{Int}(\mathcal{C}^{uu}(\varphi^{N^{uu}}_{\mu,\nu}(x))), \quad |(d\varphi^{N_3}_{\mu,\nu})_x u| \geq \rho_3 |u|$$

where $\rho_3 > 1$ is some constant.

See the proof in Appendix A.

We now give the proof of Assertion 4.1.

Proof of Assertion 4.1 . Let $\mu > 0$ be small and fixed, and D be the same box containing $p_{\mu,0}$ as above. By Lemma 3.1, there exist an integer $m_4 > 0$ and a small segment $L_{\mu,0} \subset W^u(q_{\mu,0})$ containing the quasi-transverse intersection $s_{\mu,0}$ such that, for any $m \geq m_4$, $\varphi^m_{\mu,0}(L_{\mu,0}) \cap D$ has a segment, denoted by $L^m_{\mu,0}$ as in Figure 4.4, containing $\varphi^m_{\mu,0}(s_{\mu,0})$ which is vertical through D and its tangent vector is close to $\mathbb{E}^{uu}_{p_{\mu,0}}$.

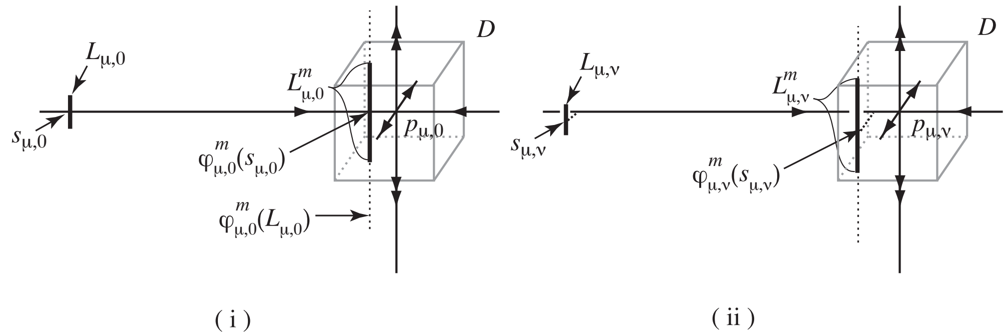

FIGURE 4.4

Let $s_{\mu,\nu}$ be a point in $W^u(q_{\mu,\nu})$ converging to to $s_{\mu,0}$ as $\nu \to 0$. For any given $m \geq m_4$, there exists an $\varepsilon = \varepsilon(m) > 0$ such that

- $\varphi^m_{\mu,\nu}(s_{\mu,\nu}) \in \mathrm{Int}(D)$ for any $\nu \in (-\varepsilon, \varepsilon)$;
- the continuation $L^m_{\mu,\nu}$ based on $L^m_{\mu,0}$ is also vertical through D, and its tangent vector is closed to $\mathbb{E}^{uu}_{p_{\mu,\nu}}$.

By using the condition of crossing bifurcation and the intermediate value theorem, one can find a large integer $n_4 > 0$ and subinterval $J_{\mu,n} \subset (-\varepsilon, \varepsilon)$ for a given $n \geq n_4$ such that, for any $\nu \in J_{\mu,n}$, $\Sigma^n_{\mu,\nu}$ intersects transversely with $W^{ss}(p_{\mu,\nu})$ at the point $t^n_{\mu,\nu}$ near $s_{\mu,\nu}$. Then, Lemma 4.2-4.4 imply that $\varphi^N_{\mu,\nu}$ satisfies conditions

(B1) and (B2) in Subsection 3.2, where $N := m + n + k$ for any $n \geq \max\{n_3, n_4\}$, $k \geq k_3$ and $m \geq \max\{m_3, m_4\}$. From the above construction, also using the intermediate value theorem, one can find a subinterval $J'_n \subset J_n$ such that the other geometric configurations of (B3) and (B4) hold for $\varphi^N_{\mu,\nu}$ with any $\nu \in J'_n$. The proof is now complete. □

4.2. **Proof of Theorem 2.8.** To detect non-connected heterodimensional cycles, we use the blender distinctive property in Subsection 3.2 and the C^1-creation lemma in Subsection 3.3 in the proof of next result.

Proposition 4.5. *Let $\mathcal{O}$ be the same open subset of parameter space $\mathbb{R}^2$ as in Assertion 4.1. There exist neighborhood $\mathcal{V}_{\mathcal{O}}$ of $\{\varphi_{\mu,\nu} : (\mu,\nu) \in \mathcal{O}\}$ in $\mathrm{Diff}^1(M)$, and dense subset $\mathcal{D}_{\mathcal{O}}$ in $\mathcal{V}_{\mathcal{O}}$ such that every $\psi \in \mathcal{D}_{\mathcal{O}}$ has a non-connected heterodimensional cycle associated with p_ψ and q_ψ.*

Proof. From Assertion 4.1, for any $(\mu,\nu) \in \mathcal{O}$, $\varphi_{\mu,\nu}$ has the saddle fixed point $q_{\mu,\nu}$ with $\mathrm{index}(q_{\mu,\nu}) = 1$, and $\varphi^N_{\mu,\nu}$ has a blender in D containing the other saddle fixed point $p_{\mu,\nu}$ with $\mathrm{index}(p_{\mu,\nu}) = 2$, where N is the same integer and D is the same box as in Assertion 4.1. Moreover, we have the same continuation $L^m_{\mu,\nu} \subset W^u(q_{\mu,\nu})$ as in the proof of Assertion 4.1, which is vertical through D to the right of $W^{ss}_{\mathrm{loc}}(p_{\mu,\nu})$. By Lemma 3.3, any vertical strip along $L^m_{\mu,\nu}$, for which tangent space is contained in the interior of the cone field $\mathcal{C}^u$, has an intersection with the closure of $W^{ss}(p_{\mu,\nu})$. Note that the blender distinctive property in Lemma 3.3 is a C^1 robust property, see [6, §6.2.2]. On the other hand, the intersection of $W^u(p_{\mu,\nu})$ and $W^s(q_{\mu,\nu})$ is transverse for any $(\mu,\nu) \in \mathcal{O}$, which is also C^1 robust. Therefore, any diffeomorphism φ in a C^1 neighborhood $\mathcal{V}(\varphi_{\mu,\nu})$ of $\varphi_{\mu,\nu}$, we have

$$\mathrm{Cl}(W^{ss}(p_\varphi)) \cap W^u(q_\varphi) \neq \varnothing, \quad W^u(p_\varphi) \cap W^s(q_\varphi) \neq \varnothing.$$

We now set

$$\mathcal{V}_{\mathcal{O}} := \bigcup_{(\mu,\nu)\in\mathcal{O}} \mathcal{V}(\varphi_{\mu,\nu}).$$

For any $\varphi \in \mathcal{V}_{\mathcal{O}}$, by Corollary 3.4, one can get ψ arbitrarily C^1-close to φ which has saddle fixed points $p_\psi = p_\varphi$ and $q_\psi = q_\varphi$, and satisfies

$$W^{ss}(p_\psi) \cap W^u(q_\psi) \neq \varnothing, \quad W^u(p_\psi) \cap W^s(q_\psi) \neq \varnothing.$$

That is, ψ has a heterodimensional cycle associated with p_ψ and q_ψ. From the setting, it is trivial that the heterodimensional cycle is non-connected. □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Let $\{\varphi_{\mu,\nu}\} \subset \mathrm{Diff}^3(M)$ be a two-parameter family with $\varphi_{0,0} = \varphi$ which is given in Theorem 2.8. For any $\mu < 0$ and $\nu \in I$, $p_{\mu,\nu}$ and $q_{\mu,\nu}$ are homoclinically related. In fact, $p_{\mu,\nu}$ and $q_{\mu,\nu}$ are saddle fixed points which have the same indices: $\mathrm{Index}(p_{\mu,\nu}) = \mathrm{Index}(q_{\mu,\nu}) = 1$. Moreover, the transversality holds for not only $W^u(p_{\mu,\nu})$ and $W^s(q_{\mu,\nu})$ but also $W^s(p_{\mu,\nu})$ and $W^u(q_{\mu,\nu})$. Therefore, every heteroclinic cycle between $p_{\mu,\nu}$ and $q_{\mu,\nu}$ is robust for $\mu < 0$. So, there exists a C^1-neighborhood, denoted by $\mathcal{U}$, of $\{\varphi_{\mu,\nu} : \mu < 0, \nu \in I\}$ such that $\mathrm{Cl}(\mathcal{U}) \ni \varphi_{0,0}$ and every $\psi \in \mathcal{U}$ also has two saddle fixed points p_ψ and q_ψ which are homoclinically related.

From the setting, for any small $\mu > 0$, $\varphi_{\mu,\nu}$ has a saddle fixed point $p_{\mu,\nu}$ with $\mathrm{Index}(p_{\mu,\nu}) = 2$ and saddle periodic point $\tilde{p}_{\mu,\nu}$ with period-two of $\mathrm{Index}(\tilde{p}_{\mu,\nu}) = 1$

such that $W^u(p_{\mu,\nu})$ and $W^s(\tilde{p}_{\mu,\nu})$ have a transverse intersection containing the $\varphi^2_{\mu,\nu}$-invariant segment connecting $p_{\mu,\nu}$ and $\tilde{p}_{\mu,\nu}$. Moreover, $W^u(\tilde{p}_{\mu,\nu})$ intersects transversely with $W^s(q_{\mu,\nu})$. By Lemma 3.1, $W^u(\tilde{p}_{\mu,\nu})$ contains a segment which is mapped arbitrarily C^1-close to the same segment $L^m_{\mu,\nu} \subset W^u(q_{\mu,\nu})$ as in the proof of Proposition 4.5. Hence, same as in Proposition 4.5, we can get

$$\mathrm{Cl}(W^{ss}(p_{\mu,\nu})) \cap W^u(\tilde{p}_{\mu,\nu}) \neq \varnothing, \quad W^u(p_{\mu,\nu}) \cap W^s(\tilde{p}_{\mu,\nu}) \neq \varnothing.$$

Also, by Corollary 3.4, one can get $\psi \in \mathcal{V}(\varphi_{\mu,\nu})$ arbitrarily C^1-close to $\varphi_{\mu,\nu}$ which has saddle fixed points $p_\psi = p_{\mu,\nu}$ and $\tilde{p}_\psi = \tilde{p}_{\mu,\nu}$, and ψ has the heterodimensional cycle which is ψ^2-connected between p_ψ and $\tilde{p}_\psi$. By a Díaz and Rocha's result in [13, Theorem A], we can get an open set $\mathcal{V}(\psi) \subset \mathcal{V}(\varphi_{\mu,\nu})$ arbitrarily C^1-close to ψ such that, for every $\tilde{\psi} \in \mathcal{V}(\psi)$, the homoclinic class of $p_{\tilde{\psi}}$ and that of $\tilde{p}_{\tilde{\psi}}$ coincide and have strongly partially hyperbolic structure. Meanwhile, since $\tilde{p}_{\tilde{\psi}}$ and $q_{\tilde{\psi}}$ are homoclinically related, the homoclinic class also contains $q_{\tilde{\psi}}$. We now get an open set $\mathcal{V}$ as the union of all $\mathcal{V}(\psi)$ with respect to ψ, which contains a dense subset $\mathcal{D} := \mathcal{D}_{\mathcal{O}} \cap \mathcal{V}$ where $\mathcal{D}_{\mathcal{O}}$ is given in Proposition 4.5. □

Appendix A. Proofs of Lemmas 4.2-4.4

Before giving the proofs, we should remark that the invariant set of D for which we construct cone fields in the lemmas contains $p_{\mu,\nu}$ but *not* $q_{\mu,\nu}$. That is, the invariant set dose *not* contain orbits which can stay in a small neighborhood of $q_{\mu,\nu}$ for an arbitrarily long time. Therefore, the proofs of lemmas are straightforward even though these are technical. In fact, procedures for proof of Lemma 4.2 and 4.3 are essentially same. So we first give the proof of Lemma 4.3.

Proof of Lemma 4.3. Set $\mu > 0$, $n \geq n_2$, $\nu \in J_{\mu,n}$ and $k \geq k_1$. For every $x \in D$, as illustrated in Figure 4.3, we define

$$\mathcal{C}^u(x) = \mathcal{C}^u(x; \theta^u) = \{u \in T_x D \ : \ \angle(u, (\mathbb{E}^{ss}_{p_{\mu,\nu}})^\perp) \leq \theta^u\},$$

where θ^u is a small constant satisfying the following condition: for every $x \in \varphi^{-k}_{\mu,\nu}(U(l_{\mu,\nu})) \cap D$ and $u \in \mathcal{C}^u(x; \theta^u)$,

$$(d\varphi^k_{\mu,\nu})_x u \in \mathcal{C}_1(\varphi^k_{\mu,\nu}(x)).$$

Note that, from the condition of around $p_{\mu,\nu}$, $\mathbb{E}^u_{p_{\mu,\nu}} = \{x^c = 0\}$ dominates $\mathbb{E}^{ss}_{p_{\mu,\nu}}$ when $\mu > 0$. This implies that there exists $k_2 \geq k_1$ such that if $k \geq k_2$ then, for any $z \in \varphi^{-k+k_2}_{\mu,\nu}(D) \cap U(\tilde{l}_{\mu,\nu})$ and $w \in \mathcal{C}_2(z)$,

$$|(d\varphi^{k-k_2}_{\mu,\nu})_z w| \geq |w|.$$

Let γ be the contracting ratio between $u \in \mathcal{C}^u$ and $(d\varphi^{n+k_2}_{\mu,\nu})u$, i.e.,

$$\gamma = \gamma(n, k_2) = 2 \sup \left\{ \frac{|u|}{|w_2|} \ : \ u \in \mathcal{C}^u(x),\ w_2 = (d\varphi^{n+k_2}_{\mu,\nu})_x u \right. \\ \left. \text{where } x \in \varphi^{-k_2}_{\mu,\nu}(U(l_{\mu,\nu})) \cap D \right\}. \tag{A.1}$$

Using this constant, we set the integer $m_2 > m_1$ satisfying as follows: if $m \geq m_2$ then

$$(d\varphi^m_{\mu,\nu})_z w \in \mathrm{Int}(\mathcal{C}^u(\varphi^m_{\mu,\nu}(z))), \quad \left|(d\varphi^m_{\mu,\nu})_z w\right| \geq \gamma\, |w| \tag{A.2}$$

for every $z \in \varphi_{\mu,\nu}^{-m}(D) \cap U(\tilde{l}_{\mu,\nu})$ and $w \in \mathcal{C}_2(z)$.

Now, set $N^u := n + k + m$ for $k \geq k_2$, $m \geq m_2$ and $n \geq n_2$. The above construction implies that $\varphi_{\mu,\nu}(D) \cap D$ contains a connected component A satisfying $p_{\mu,\nu} \in \mathrm{Int}(A)$. Moreover, $\varphi_{\mu,\nu}^{N^u}(D) \cap D$ contains a connected component B such that

$$B = \varphi_{\mu,\nu}^{m+n}(\varphi_{\mu,\nu}^{k}(D) \cap U(l_{\mu,\nu})) \cap D.$$

For any $x \in \varphi_{\mu,\nu}^{-1}(A)$, we have $\varphi_{\mu,\nu}(x) \in D$. So, by the linearization at $p_{\mu,\nu}$, (4.3) in the lemma is trivial. On the other hand, for $x \in \varphi_{\mu,\nu}^{-N^u}(B)$, by (A.2), we have that for nonzero $u \in \mathcal{C}^u(x)$,

$$(d\varphi_{\mu,\nu}^{N^u})_x u \in \mathrm{Int}(\mathcal{C}^u(\varphi_{\mu,\nu}^{N^u}(x)))$$

and

$$\left|(d\varphi_{\mu,\nu}^{N^u})_x u\right| \geq \gamma \, |v| = \gamma \left|(d\varphi_{\mu,\nu}^{k-k_2})_z w_2\right| \geq \gamma |w_2| \geq 2 \, |u| \, ,$$

where $v = (d\varphi_{\mu,\nu}^{k+n})_x u$, $w_2 = (d\varphi_{\mu,\nu}^{k_2+n})_x u$ and $z = \varphi_{\mu,\nu}^{k_2+n}(x)$. The last inequality is obtained from (A.1). This completes the proof. □

Proofs of Lemma 4.2 and 4.4. The procedure of proof of Lemma 4.2 is essentially same as that of Lemma 4.3. Moreover, the proof of Lemma 4.4 is easier than that of Lemma 4.3. In fact, evaluation of expanding ratio of forward images for vectors in $\mathcal{C}^{uu}$ is easier than that of Lemma 4.3, because these images are away from the center stable direction near $q_{\mu,\nu}$. □

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