On transition matrices

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Abstract

In the Conley index theory, transition matrices are used to detect bifurcations of codimension one connecting orbits in a Morse decomposition of an isolated invariant set. Here we shall give a new axiomatic definition of the transition matrix in order to treat several existing formulations of transition matrices in a unified manner.

1 Introduction

In the Conley index theory, the transition matrix is used to detect change of connecting orbit structure of the Morse decomposition of isolated invariant set when parameters are varied. Such a situation is illustrated in Figure 1 where one intuitively expects that there should exist a structurally unstable (codimension 1) connecting orbit from $M(3)$ to $M(2)$ at some parameter value λ_* in order for the system to change the orbit structure from $\lambda = 0$ to $\lambda = 1$. The transition matrix obtained for this example is a 3×3 upper triangular invertible matrix given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ whose off-diagonal (2,3)-entry being non-zero implies the existence of connecting orbit from $M(3)$ to $M(2)$ at some value of parameter $\lambda_* \in (0, 1)$.

The idea of transition matrix is originally due to Conley, whose formulation was explicitly given by Reineck[9] for the first time. Since then there have been several different formulations as well as generalizations of transition matrices.

In this paper we shall compare those formulations and give an axiomatic definition of the transition matrix and its existence in order to treat those previously given formulations in a unified manner. We shall also discuss a generalization of the new formulation of the transition matrix, based on a joint work with H. Oka; the details will be published elsewhere.

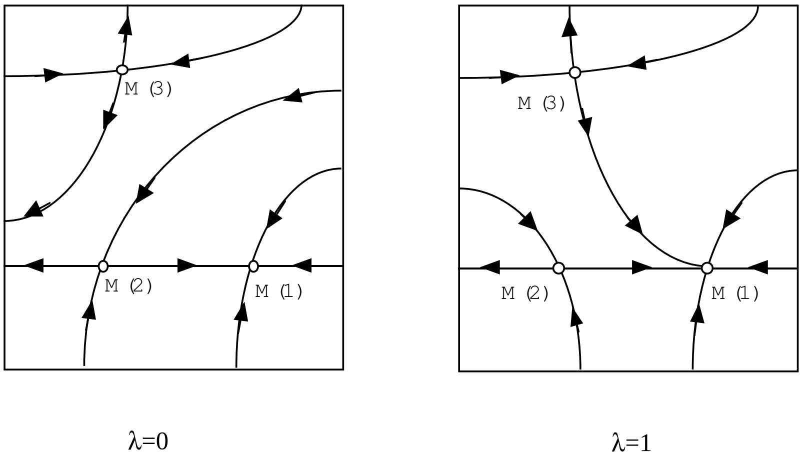

Figure 1: Change of connecting orbits.

2 Preliminaries

Here we shall very briefly review basic definitions from the Conley index theory. For details, we refer to Conley[1], Salamon[10], Mischaikow[8] and references therein.

For a topological flow $\varphi : X \times \mathbb{R} \to X$ on a locally compact metrix space, an *isolated invariant set* is a compact invariant subset of X, which is maximally invariant in its compact neighborhood, and which lies in its interior. The compact neighborhood is called its *isolating neighborhood* and is said to isolate its maximal invariant set. Given an isolating neighborhood N of an isolated invariant set S, its *exit set* L is a closed subset of N that satisfies the following: (1) L is positively invariant in N; (2) $N \setminus L$ isolates S; (3) an orbit leaving N must leave through L. The pair (N, L) is called an *index pair*. It can be shown that the homotopy type of the quotient space N/L depends only on the isolated invariant set S and is independent of the choice of its index pair, and hence the *homotopy Conley index* of S, denoted $h(S)$, is defined by the homotopy type of N/L, whereas the *homology Conley index* $CH_*(S)$ is defined as the homology $H_*(N/L, [L])$. Since the Conley index is topologically invariant, it can be used to distinguish topologically different flows.

The *Morse decomposition* was introduced in order to capture more detailed structure of an isolated invariant set S. It is a collection of finitely many disjoint isolated invariant subsets $M(p)$ of S, called a *Morse component*, indexed by a finite partially ordered set $(P, <)$ that satisfies the following condition: for any orbit $\gamma \not\subset \cup_{p\in P} M(p)$, there exist $p, q \in P$ with $p < q$ such that $\alpha(\gamma) \subset M(q)$ and $\omega(\gamma) \subset M(p)$. Such an orbit is called a *connecting orbit* from $M(q)$ to $M(p)$, all of which forms the set $C(p, q)$. As a consequence, given a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (P, <)\}$ of S, we have $S = [\cup_{p\in P} M(p)] \cup [\cup_{p<q} C(p, q)]$. It is important to notice that if $\mathcal{M}(S)$ is a Morse decomposition of S with the order $<$, then so is with an extension of $<$. If it is necessary to distinguish different partial orders to the same index set P, we may indicate the order as $\mathcal{M}_<(S)$. The *flow-defined order* $<^F$ is the minimal partial order for $\mathcal{M}_{<^F}(S)$ to be a

Morse decomposition of S.

Franzosa[2] introduced the notion of connection matrix, which is a very convenient tool to see how the Morse components are connected. Given a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (P,<)\}$, a *connection matrix* Δ is a degree -1 linear map from $\oplus_{p\in P} CH_*(M(p))$ to itself satisfying: (1) it is strictly upper triangular, i.e. $\Delta(p,q) \neq 0$ implies $p < q$; (2) $\Delta^2 = 0$; (3) $\mathrm{Ker}\Delta/\mathrm{Im}\Delta$ is isomorphic to $CH_*(S)$. In general, a connection matrix for any Morse decomposition $\mathcal{M}_<(S)$ exists, but it is not unique. See Franzosa[2] for details.

If one knows that a connection matrix Δ has a non-zero off-diagonal entry $\Delta(p,q)$ with respect to the flow-defined order, it means that there exist p_i $(i = 1,\dots,k)$ with $p = p_1,\ q = p_k$ and $C(p_i, p_{i+1}) \neq \emptyset$ for any $i = 1,\dots,k-1$. If the order $<$ is admissible, namely it is an extension of $<^F$, then the same conclusion holds provided that the (p,q)-off diagonal entry is non-zero for every connection matrix of the Morse decomposition $\mathcal{M}_<(S)$. This is not too hard to show, because any connection matrices of a given Morse decomposition share common algebraic properties as given above, and many of the information about connection matrix entries can be obtained only using these algebraic properties.

For application, we often need to study a family of flows φ_λ on a space X with the parameter $\lambda \in \Lambda$ and changes of dynamical structure as parameters are varied in Λ. In order to speak about Conley indices and relevant notion for parameter families, it is convenient to extend the flow to the parametrized flow $\Phi : X \times \Lambda \times \mathbb{R} \to X \times \Lambda$ given by $\Phi(x,\lambda,t) = (\varphi_\lambda(x,t),\lambda)$. In this situation, we say that a family of isolated invariant sets $\{S_\lambda\}$ continues over Λ if S_Λ defined as $\cup_{\lambda\in\Lambda} S_\lambda \times \{\lambda\}$ is an isolated invariant set of the parametrized flow. If this is the case, there exists a natural isomorphism called the continuation isomorphism, from $CH_*(S_\lambda)$ to $CH_*(S_\Lambda)$ induced by the natural inclusion of the index pairs (N_λ, L_λ) to (N_Λ, L_Λ). Similarly, we can speak about continuation of Morse decompositions $\mathcal{M}(S_\lambda)$: $\mathcal{M}(S_\lambda)$ with a partial-order $<_\lambda$ is said to continue over Λ if $M_\Lambda(p) = \cup_{\lambda\in\Lambda} M_\lambda(p) \times \{\lambda\}$ $(p \in P)$ forms a Morse decomposition of the parametrized flow with respect to a partial-order $<_\Lambda$. Clearly, the order $<_\Lambda$ is an extension of $<_\lambda$ for any $\lambda \in \Lambda$. In prticular, when $<_\lambda$ is the flow-defined order for each λ, the minimal order $<$ for which the collection $\{M_\Lambda(p) \mid p \in P\}$ becomes a Morse decomposition of Φ is called the flow-defined order over Λ and is denoted by $<^F_\Lambda$.

3 Axiomatic formulation of transition matrix

Similarly to connection matrices that detect existence of connecting orbits for a given Morse decomposition, the transition matrix gives algebraic information about changes of connecting orbits in a one-parameter family of flows. There have been several different formulation of transition matrices: singular transition matrix introduced in Reineck[9]; topological transiton matrix given by McCord-Mischaikow[5]; algebraic transition matrix formulated by Franzosa-Mischaikow[3].

When Λ is an arc, say $[0,1]$, and X is a manifold, the singular transition matrix is given as a connection matrix of the extended flow on $X \times \Lambda$ given by the coupled

ODE $\dot{x} = f(x, \lambda)$, $\dot{\lambda} = g(\lambda)$ where $g(\lambda)$ satisfies $g(0) = g(1) = 0$ and $-1 \ll g(\lambda) < 0$ for $\lambda \in (0,1)$. For instance one can take $g(\lambda) = \varepsilon\lambda(1-\lambda)$ with small $\varepsilon > 0$. The dynamics of λ is introduced artificially, hence called an artificial parameter slow drift. For this extended flow, there exists a natural Morse decomposition consisting of $M_0(p)$'s and $M_1(p)$'s with a partial order $<$ given by $<_0$ and $<_1$ together with $(p,0) < (q,1)$ for any $p, q \in P$, and therefore the corresponding connection matrix $\tilde{\Delta}$ is a block-triangular matrix $\begin{pmatrix} \Delta_0 & T \\ O & \Delta_1 \end{pmatrix}$ where Δ_i is a connection matrix for each $\lambda = i = 0, 1$. From algebraic properties of the connection matrix $\tilde{\Delta}$, one can show that the matrix T is considered as a degree 0 isomorphism from $\oplus_{p\in P} CH_*(M_1(p))$ to $\oplus_{p\in P} CH_*(M_0(p))$ which is upper-triangular and satisfies $\Delta_0 T + T\Delta_1 = 0$. The last equality comes from $\tilde{\Delta}^2 = 0$. If $<$ is the flow-defined order over Λ, then $T(p,q) \neq 0$ for a transition matrix T implies that there exist $\{\lambda_i\}_{i=1,\dots,k}$ and $\{p_i\}_{i=1,\dots,k+1}$ such that $p = p_1$, $q = p_{k+1}$ and $C(p_i, p_{i+1}) \neq \emptyset$ at $\lambda = \lambda_i$ for $i = 1, \dots, k$. Moreover, since the parameter slow drift moves monotonically, $\{\lambda_i\}$ can be chosen monotonically decreasing. If $<$ is an extension of the flow-defined order, then the same conclusion holds provided that the (p,q)-entry being non-zero for all possible transition matrices.

Although this formulation of transition matrix has been applied successfully to many different bifurcation problems, there is a serious problem in that it is unclear how the transition matrix depends on the choice of the parameter slow drift. In particular, one does not know the 'transitivity' of transition matrices, namely, whether the composition of one transition matrix from $\lambda = 2$ to 1 and another from $\lambda = 1$ to 0 gives a transition matrix from $\lambda = 2$ to 0. In order to formulate the singular transition matrix as less dependent of the choice of parameter slow drifts as possible, McCord-Mischaikow[6] considered the set of all extended flows with possible parameter slow drifts, and gave a more precise definition of the singular transition matrix as a possible limit of the upper-right block of a connection matrice of the extended flow when the size of the parameter slow drift goes to 0 in supremum norm. With this definition, they also succeeded in proving the equivalence of the singular transition matrix and the topological transition matrix which will be explained shortly, when both are defined. In particular, the transitivity holds in that case, because the topological transition matrix always satisfies the transitivity. In [4], this equivalence is used to show the existence of infinitely many connecting orbits in a slowly varying Hamiltonian system.

The topological transition matrix was introduced by McCord-Mischaikow [5] in order to formulate the transition matrix independent of the artificial parameter slow drift. For this purpose, they have assumed that there is no connecting orbit in the Morse decomposition at the boundary of the parameter arc $\Lambda = [0,1]$, and therefore $S_i = \sqcup_{p\in P} M_i(p)$, hence $CH_*(S_i) = \oplus_{p\in P} CH_*(M_i(p))$ at $\lambda = i = 0, 1$. Recall that we also have the global continuation isomorphism $(F_{01})_* : CH_*(S_1) \to CH_*(S_0)$ as well as the sum of local continuation isomorphisms $\oplus_{p\in P}(F_{01})_* : \oplus_{p\in P} CH_*(M_1(p)) \to \oplus_{p\in P} CH_*(M_0(p))$. These four isomorphisms form a square diagram that does not commute in general, and the topological transition matrix is defined as a matrix representation of the global continuation isomorphism with the choice of bases so that the local continuation isomorphisms become identity. Since it is essentially the global continuation isomorphism which clearly satisfies the transitivity, one immediately sees that the topological transiton matrix T also satisfies the transitivity as well. Moreover, it is an upper-triangular isomorphism that trivially satisfies the equality $\Delta_0 T + T\Delta_1 = 0$, since the connection matrices at the boundary of parameter arc are zero due to the non-existence of connecting orbits at $\lambda = 0, 1$. From the upper-triangularity of the transition matrix, one can prove that a non-zero off-diagonal entry implies the existence of codimension one connecting orbit that appears when the connecting orbit structure changes as the parameter λ is varied from $\lambda = 1$ to 0.

Of course the major drawback of the topological transition matrix is that it is not defined when there are connecting orbits at the boundary of the parameter space. Franzosa-Mischaikow[3] intended to overcome this difficulty by carrying out purely algebraic construction of transition matrices, which is called the algebraic transition matrix. For this, they needed to make some technical assumption on the partial order, with which they have proven the existence of the transition matrix as an upper-triangular degree 0 isomorphism which is a similarity isomorphism between the boundary connection matrices $\Delta_i,\ i = 0, 1$. As we shall see below, one crucial condition is missing in their definition in relation to the continuation isomorphism, although they have indeed used it in their existence proof.

Now we are in position to give our formnulation of the transition matrix: Let α be an arc in a path-connected space Λ that joins λ_0 and λ_1. Assume that we have Morse decompositions $\mathcal{M}(S_\lambda)$ that continue over the parameter arc α with an addmissible order $<$. Let Δ_i be connection matrices at the boundary of α, and, for each $\lambda = i = 0, 1$, let $\mathcal{C}\Delta_i = \oplus_{p\in P} CH_*(M_i(p))$ be the chain complex with Δ_i being its boundary map.

Definition 3.1 *Under the above situation, a transition matrix is a chain map $T : \mathcal{C}\Delta_1 \to \mathcal{C}\Delta_0$ which is an isomorphism, is upper-triangular with respect to $<$, and whose induced homomorphism $T_* : CH_*(S_1) \to CH_*(S_0)$ is in fact the global continuation isomorphism. The set of all such chain maps, if it exists, is denoted $\mathcal{T}_\alpha(\Delta_0, \Delta_1)$. Similarly, the union of $\mathcal{T}_\alpha(\Delta_0, \Delta_1)$ over all possible boundary connection matrices is denoted $\mathcal{T}_\alpha(\lambda_0, \lambda_1)$.*

With this definition, the main result of this paper is the following existence theorem.

Theorem 3.2 *The set $\mathcal{T}_\alpha(\lambda_0, \lambda_1)$ is not empty.*

Notice that our definition of transition matrix is similar to the algebraic transition matrix defined by [3]. However, there are two major differences: Firstly, the algebraic transition matrix given in [3] does not explicitly give relation to the global continuation isomorphism which is crucial for extending the topological transition matrix to the situation where boundary connection matrices are non-trivial. Secondly, dealing with a transition matrix as a chain map rather than a similarity isomorphism is important in relation to the topological transition matrix and for further generalization to families with more than one parameters. We shall discuss this in the next section.

The proof of the above theorem is not hard, once it is formulated. We employ the singular transition matrix construction given in [6], in particular use the fact that the constructed transition matrix is upper-triangular with respect to the drift-partial order which is even weaker than the flow-defined order, hence than the partial order $<$. One can then easily verify the remaining properties.

4 Generalization to multiparameter families

Given an arc α in the parameter space, one has connection matrices Δ_i at λ_i, $i = 0, 1$, the boundary points of α, as well as a transition matrix $T \in \mathcal{T}_\alpha(\Delta_0, \Delta_1)$. From these data, one can construct a new chain complex as the mapping cone $\mathcal{C}T$ of the chain map $T : \mathcal{C}\Delta_1 \to \mathcal{C}\Delta_0$ whose boundary map Δ_T is given by $\begin{pmatrix} -\Delta_0 & T \\ O & \Delta_1 \end{pmatrix}$. One can then prove that the induced homology group $H_*(\mathcal{C}T)$ is in fact isomorphic to $CH_{*+1}(S_\alpha)$, the homology Conley index of $S_\alpha = \cup_{\lambda \in \alpha} S_\lambda$.

Suppose the parameter space Λ is simply connected and we choose two arcs α and α' joining the same points λ_0, λ_1 that bound a 2-disk D. Then we can extend the definiton of transition matrix so that it can be used to detect a codimension two connecting orbit that exists at a parameter value in the 2-disk as follows: Let A be a degree $+1$ linear map from $\mathcal{C}\Delta_1$ to $\mathcal{C}\Delta_0$ which is upper-triangular with respect to an addmissible partial order over D and is a chain homotopy between T and T'. From the last condition, A induces a chain map $\tilde{A}$ from $\mathcal{C}T'$ to $\mathcal{C}T$. We moreover require that the induced map $\tilde{A}_* : CH_{*+1}(S_{\alpha'}) \to CH_{*+1}(S_\alpha)$ is equivalent to the global continuation isomorphism $(F_{\alpha\alpha'})_*$. We can prove that such a map A indeed exists for some choice of connection matrices and transition matrices. As a consequence, its non-zero off-diagonal (p, q)-entry implies the existence of a sequence of codimension two connecting orbits in the interior of the parameter 2-disk D, and therefore such maps can be considered as generalization of transition matrices for multiparameter families of flows. This result is obtained by a joint work with H.Oka, and the details will appear elsewhere.

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