A degenerate singularity generating geometric Lorenz attractors

Freddy Dumortier
Departement Wiskunde, Limburgs Universitair Centrum
Universitaire Campus, B-3590 Diepenbeek, BELGIUM

Hiroshi Kokubu
Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606-01, JAPAN

and

Hiroe Oka
Department of Applied Mathematics and Informatics
Faculty of Science and Technology, Ryukoku University
Seta, Otsu 520-21, JAPAN

January 20, 1995

Abstract

A degenerate vector field singularity in $\mathbb{R}^3$ can generate a geometric Lorenz attractor in an arbitrarily small unfolding of it. This enables us to detect Lorenz-like chaos in some families of vector fields, merely by performing normal form calculations of order 3.

1 Introduction

The local bifurcation theory for vector fields deals with changes of dynamical structure of a vector field germ under perturbation. The Hartman-Grobman Theorem [Palis and de Melo 1982] tells us that a hyperbolic vector field singularity is locally structurally stable, namely, it has the same dynamical structure as a nearby vector field singularity does — no bifurcation occurs after perturbation. Singularities other than hyperbolic ones, on the contrary, do undergo bifurcations and hence they are called "degenerate singularities".

Bifurcations appearing from such degenerate singularities strongly depend on the singularity itself. A singularity having a simple zero eigenvalue in its linear part as well as non-degenerate higher order terms generates a pair of equilibria after perturbation, which is called the saddle-node bifurcation. A singularity having one pair of pure imaginary eigenvalues in the linear part together with non-degenerate nonlinear terms gives rise to a limit cycle bifurcating off from the singularity, which is known as the Andronov-Hopf bifurcation (e.g. [Dumortier 1991]). More complicated bifurcations occur from more degenerate singularities, by which a homoclinic orbit, an invariant torus, or chaotic dynamics can be generated [Guckenheimer and Holmes 1983]. Those degenerate singularities have been studied extensively but many important cases are still left for further investigation.

The purpose of this paper is to show that even a strange attractor can be generated from certain degenerate singularities. Namely, we shall prove the following:

Theorem 1.1. *Let v be a vector field singularity in $\mathbb{R}^3$ whose 3-jet is given by*

$$y\frac{\partial}{\partial x} - x^3\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z} \qquad ((x, y, z) \in \mathbb{R}^3).$$

Then there exists an arbitrarily small unfolding of the singularity which contains a geometric Lorenz attractor.

Here a geometric Lorenz attractor ([Guckenheimer 1973], [Guckenheimer and Williams 1979], [Williams 1979]) is a mathematically formulated geometric model of the Lorenz attractor ([Lorenz 1963]) which is a strange attractor observed by computer simulation of the following ordinary differential equation:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{1.1}$$

with $\sigma = 10, r = 28, b = \frac{8}{3}$. The above theorem shows that an unfolding of certain degenerate singularities contains a (miniature of) strange attractor which satisfies the definition of geometric Lorenz attractors given by Guckenheimer and Williams. More precise statement is given in §3. Important is that the existence of such a geometric Lorenz attractor in an unfolding can be verified by a simple normal form calculation of equations up to terms of order 3, and therefore it may be possible to use the result as a criterion for the existence of "Lorenz-like chaos" in given systems — in a similar way as the Andronov-Hopf bifurcation theorem has been used for a criterion to detect an oscillatory motion.

For the rest of this paper, we shall give a proof of the above theorem. In §2, the singularity is heuristically introduced from the original Lorenz equation (1.1) according to [Ushiki, Oka and Kokubu 1984], and in §3, the main result is stated in a more precise manner by specifying the necessary unfolding terms. In §4, we shall give several additional properties of the singularity. §5 is devoted to a preliminary rescaling argument by which the unfolding is reduced to a certain type of homoclinic bifurcation problem. This particular homoclinic orbit is called an 'inclination-flip' homoclinic orbit studied by [Yanagida 1987], [Kisaka, Kokubu and Oka 1992], and [Rychlik 1990]. The precise definition will be given in §6 together with a theorem due to [Rychlik 1990] which plays an important role for our result: He showed that a symmetric pair of such inclination-flip homoclinic orbits with a certain eigenvalue condition can generate a geometric Lorenz attractor in its perturbation. In order to apply Rychlik's theorem to our case, we need to perturb our system in order to verify the eigenvalue condition, and therefore we need to study the persistence of inclination-flip homoclinic orbits. This task will be carried out in §7 by introducing a new Melnikov-like integral adapted to the inclination-flip homoclinic orbits. In §8 we compute the integrals for our unfolding family, and finally in §9 we conclude the proof of our main result. We discuss several related results in §10.

Acknowledgement

This work has been done while HK was visiting the Limburgs Universitair Centrum (Belgium) and the Center for Dynamical Systems and Nonlinear Studies in the Georgia Institute of Technology (U.S.A.) in 1992. He would like to thank these institutes for their warm hospitality, and also thank the Kyoto University Foundation for its financial support which enabled to make these visits. HO is supported in part by the Human·Science·Religion Research-Aid Fund from Ryukoku University.

2 The Lorenz equation and its scaling limit

We start from the original Lorenz equation (1.1) with different notation for convenience:

$$\begin{aligned} X' &= \sigma(Y - X), \\ Y' &= rX - Y - XZ, \\ Z' &= -bZ + XY, \end{aligned} \quad \left(' = \frac{d}{d\tau}\right),$$

and make a change of coordinates as follows ([Ushiki, Oka and Kokubu 1984]):

$$\bar{X} = \frac{X}{\sqrt{2}}, \quad \bar{Y} = \frac{\sigma}{\sqrt{2}}(Y - X), \quad \bar{Z} = \frac{\sigma}{2\sigma - b}\left(Z - \frac{X^2}{2\sigma}\right),$$

which embeds the Lorenz equation into

$$\begin{aligned} \bar{X}' &= \bar{Y}, \\ \bar{Y}' &= a\bar{X} - \bar{X}^3 + p\bar{Y} + q\bar{X}\bar{Z}, \\ \bar{Z}' &= -b\bar{Z} + \bar{X}^2. \end{aligned} \tag{2.1}$$

In fact, when

$$a = \sigma(r - 1), \quad p = -(\sigma + 1), \quad q = -(2\sigma - b),$$

then the system (2.1) is reduced back to the original Lorenz equation. Notice that the transformed equation still preserves the same symmetry as of the Lorenz equation.

Then we make further rescaling of the system as follows:

$$x = \varepsilon\bar{X}, \quad y = \varepsilon^2\bar{Y}, \quad z = \varepsilon\bar{Z}, \quad t = \frac{\tau}{\varepsilon},$$

which brings (2.1) into

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon^2 ax - x^3 + \varepsilon py + \varepsilon qxz, \\ \dot{z} &= -\varepsilon bz + x^2, \end{aligned} \quad \left(\dot{} = \frac{d}{dt}\right), \tag{2.2}$$

and therefore, taking the limit $\varepsilon \to 0$, we obtain the degenerate system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x^3, \\ \dot{z} &= x^2, \end{aligned} \tag{2.3}$$

From this, one can observe that any dynamics appearing in the original Lorenz equation including a numerically generated Lorenz attractor for the standard parameters $\sigma = 10, r = 28, b = \frac{3}{8}$ can be put into some but arbitrarily small perturbation of the degenerate system (2.3). Of course this does not show the existence of a chaotic attractor in an unfolding of the degenerate singularity (2.3), because there is no rigorous proof for the existence of chaotic attractors in the original Lorenz equation, up to now.

3 The main result

We shall give a definition of the geometric Lorenz attractor. Throughout this paper, vector fields are assumed to be in three dimension and have the symmetry given by the linear map $g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, namely, a vector field v on $\mathbb{R}^3$ is assumed to satisfy

$$v(g\mathbf{x}) = gv(\mathbf{x}), \qquad \forall \mathbf{x} = (x, y, z) \in \mathbb{R}^3.$$

We call such a vector field a *g-equivariant vector field.*

Consider a g-equivariant vector field v on $\mathbb{R}^3$ satisfying the following conditions:

(GL1)

$$v(O) = 0 \quad \text{and} \quad Dv(O) = \begin{pmatrix} \lambda^u & 0 & 0 \\ 0 & \lambda^{ss} & 0 \\ 0 & 0 & \lambda^s \end{pmatrix}$$

where

$$0 < -\lambda^s < \lambda^u < -\lambda^{ss}.$$

In particular the z-axis is the eigendirection to the stable eigenvalue λ^s and at the same time it is invariant under the symmetry g.

(GL2) There exists a rectangular cross section R transverse to the flow generated by v which intersects the z-axis, such that the rectangle R is mapped back into itself by the flow. The image of R consists of two cusp shaped regions.

(GL3) There exists a local coordinate (ξ, η) on R such that the Poincaré map $\Pi : R \to R$ generated by the flow takes the form

$$\Pi(\xi, \eta) = (\Pi_1(\xi), \Pi_2(\xi, \eta)) \tag{3.1}$$

for $\xi \neq 0$, with smooth functions Π_1, Π_2 satisfying

- $\Pi(-\xi, -\eta) = -\Pi(\xi, \eta)$;
- $\{\xi = 0\}$ corresponds to the stable manifold $W^s(O) \cap R$;
- $\Pi_1'(\xi) > \sqrt{2}$ and $\lim_{\xi \to 0} \Pi_1'(\xi) = +\infty$;

- There exists a constant c with $0 < c < 1$ such that

$$0 < \frac{\partial \Pi_2}{\partial \eta}(\xi, \eta) < c < 1 \quad \text{for} \quad \xi \neq 0$$

and

$$\lim_{\xi \to 0} \frac{\partial \Pi_2}{\partial \eta}(\xi, \eta) = 0.$$

A g-equivariant vector field v satisfying these conditions is called a *geometric Lorenz model*. Intuitively these conditions seem to fit nicely with the computer picture from the original Lorenz equation. However, two strong conditions are imposed here: one is that the Poincaré map is assumed to preserve lines $\xi =$ constant due to the form of the map (3.1), and then the other requirement is the hyperbolicity, namely, expansion in ξ-direction and contraction in η-direction. This is equivalent to say that the Poincaré map Π on R admits an invariant foliation, on each leaf of which the map is contracting while it is expanding in the transverse direction of the leaves. These strong conditions are used to reduce the study of the entire Poincaré map Π to that of the one-dimensional map $\Pi_1(\xi)$. Since the one-dimensional map is uniformly expanding and monotone increasing with a single discontinuity at $\xi = 0$, its dynamics is very well understood ([Guckenheimer and Williams 1979], [Rand 1979], [Keller 1985], [Robinson 1984]). In particular, we can conclude that the geometric Lorenz model has an attractor whose dynamics is essentially described by the reduced one-dimensional mapping Π_1. This attractor is called the *geometric Lorenz attractor*. One of the important consequences of this is that the geometric Lorenz attractor is not structurally stable, although it is C^r-persistent for large enough r. This follows from the fact that the kneading sequence determines its topological conjugacy class for the one-dimensional mappings of this type. See [Guckenheimer and Williams 1979], [Rand 1978] for details. Note also that a similar result was obtained by [Afraimovich, Bykov and Shil'nikov 1982] independently where an analogous description of the dynamics of Lorenz attractor was given based on a continuous invariant foliation. See also [Afraimovich and Pesin 1987].

Though these assumptions (GL1)-(GL3) are not easily verifiable for a given vector field in general, it is possible to show the existence of the geometric Lorenz attractor in the context of homoclinic bifurcations. In §6, we shall briefly explain an idea given by [Rychlik 1990].

Our main result of this paper is the following:

Theorem 3.1. *There exist arbitrarily small values of the parameters* $(\alpha, \gamma, A, B, C)$ *with* $\frac{1}{2} < -\frac{\gamma}{\sqrt{\alpha}} < 1$ *and* $A < 0$ *such that any three-dimensional ordinary differential equation whose third order truncation takes the form*

$$
\begin{array}{rcl}
\dot{x} & = & y, \\
\dot{y} & = & \alpha x - x^3 + Ay + Bxz + Cyz, \\
\dot{z} & = & \gamma z + x^2,
\end{array} \tag{3.2}
$$

contains a geometric Lorenz attractor.

Needless to say that the equation (3.2) gives an unfolding of the degenerate singularity having the 3-jet

$$
y\frac{\partial}{\partial x} - x^3\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}
$$

given in §1. Hereafter we call such a degenerate singularity a *Lorenz singularity.*

4 Properties of the Lorenz singularity

In this section, we give two properties of "Lorenz singularities". First we calculate the codimension among g-equivariant vector field singularities and second we show that a Lorenz singularity cannot be found among quadratic vector fields in $\mathbb{R}^3$. Both properties are proved by using the usual normal form theory (See e.g. [Vanderbauwhede 1989] or [Dumortier 1991]). Before this, let us give a precise definition of "Lorenz singularity", emphasizing that the definition is aimed at characterizing, among g-equivariant vector fields, the largest class of singularities of which we can prove (with the techniques used in this paper) that they can give birth to a geometric Lorenz attractor.

Definition 4.1. A vector field germ in $(\mathbb{R}^3, O)$ is called a *Lorenz singularity* if its 3-jet at O is C^∞-equivalent to

$$
y\frac{\partial}{\partial x} + (-x^3 + bx^2y + cyz^2)\frac{\partial}{\partial y} + (x^2 + ex^2z + fz^3)\frac{\partial}{\partial z}
$$

for some $(b, c, e, f) \in \mathbb{R}^4$.

The "normal form" for C^∞-conjugacy would be

$$
y\frac{\partial}{\partial x} + (ax^3 + bx^2y + cyz^2)\frac{\partial}{\partial y} + (dx^2 + ex^2z + fz^3)\frac{\partial}{\partial z}
$$

with $a < 0$, $d \neq 0$ and (b, c, e, f) arbitrary; but using a linear coordinate change and a linear time scale, we can normalize the coefficients $a,\ d$ to $-1,\ 1$, respectively.

In the next proposition, we will not aim at giving a precise calculation of the codimension of a Lorenz singularity, but we will only obtain an upper bound by means of the standard normal form theory.

Proposition 4.2. *The Lorenz singularities have codimension at most 7 among g-equivariant vector field germs.*

Proof.
We work on the jet space of g-equivariant vector fields:

$$H^g = H_1^g \oplus H_2^g \oplus H_3^g \oplus \cdots,$$

where H_k^g stands for the vector space consisting of all g-equivariant homogeneous vector fields of degree k. The linear part L of a Lorenz singularity v yields the image of the adjoint operator $ad(L)_2 : H_2^g \to H_2^g$ as given by

$$B_2^g = \mathrm{Im} ad(L)_2 = \mathrm{span}_{\mathbb{R}} \left\{ yz\frac{\partial}{\partial x}, xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y}, xy\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial z} \right\},$$

and hence we can take the complementary space of B_2^g in H_2^g to be

$$G_2^g = \mathrm{span}_{\mathbb{R}} \left\{ xz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z} \right\}.$$

Since the singularity v does not contain terms with $xz\frac{\partial}{\partial y}$, $yz\frac{\partial}{\partial y}$ and $z^2\frac{\partial}{\partial z}$, we have 3 codimensions from the second order part of v.

We proceed in the same way for the third order part. We obtain the complementary space

$$G_3^g = \mathrm{span}_{\mathbb{R}} \left\{ x^3\frac{\partial}{\partial y}, x^2y\frac{\partial}{\partial y}, xz^2\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, x^2z\frac{\partial}{\partial z}, z^3\frac{\partial}{\partial z} \right\},$$

where only $xz^2\frac{\partial}{\partial y}$ disappears in the singularity v.

Therefore, counting 3 more codimensions from the linear part, we conclude that the singularity v has at most codimension 7. □

Proposition 4.3. *A quadratic vector field in $\mathbb{R}^3$ cannot have a Lorenz singularity.*

Proof. The quadratic vector field Y will have a Lorenz singularity at O if and only if there exists a local diffeomorphism f with $f_*X = Y$ for some "normal form"

$$X = (y + O(2))\frac{\partial}{\partial x} + (-x^3 + yO(2)) + O(3))\frac{\partial}{\partial y} + (x^2 + O(3))\frac{\partial}{\partial z}$$

like in Definition 4.1. If $f = A(I + P)$ with A linear and $P = O(2)$, then $Y = f_* X$ will be quadratic if and only if $(I + P)_* X$ is.

Let now $\dot{u} = X(u)$ represent a Lorenz singularity in normal form and take $u = (I + Q)(v)$ with $v = (I + P)(u) = (I + Q)^{-1}(u)$, with $Q = O(2)$, $P = O(2)$ and with $Y = (I + P)_* X$ quadratic. We denote $u = (x, y, z)$ and $v = (\tilde{x}, \tilde{y}, \tilde{z})$. Then

$$\begin{aligned} Y(v) &= (I + DP)((I + Q)(v)) \cdot X((I + Q)(v)) \\ &= \begin{pmatrix} 1 + O(1) & O(1) & O(1) \\ O(1) & 1 + O(1) & O(1) \\ O(1) & O(1) & 1 + O(1) \end{pmatrix} \begin{pmatrix} \tilde{y} + O(3) \\ O(3) \\ \tilde{x}^2 + O(3) \end{pmatrix} \\ &= (\tilde{y}(1 + O(1)))\tfrac{\partial}{\partial \tilde{x}} + \tilde{y}O(1)\tfrac{\partial}{\partial \tilde{y}} + (\tilde{x}^2 + \tilde{y}O(1))\tfrac{\partial}{\partial \tilde{z}}, \end{aligned}$$

since Y is assumed to be quadratic. This last equation cannot have an isolated zero. □

5 Rescaling of an unfolding of the Lorenz singularity

When studying unfoldings of a Lorenz singularity, like the one in (3.2), we will use the following rescaling:

$$\begin{array}{lll} x = \varepsilon \bar{x}, & \alpha = \varepsilon^2 \bar{\alpha}, & A = \varepsilon \bar{A}, \\ y = \varepsilon^2 \bar{y}, & \gamma = \varepsilon \bar{\gamma}, & B = \varepsilon \bar{B}, \\ z = \varepsilon \bar{z}, & & C = \bar{C}, \\ t = \varepsilon^{-1} \bar{t}. & & \end{array} \tag{5.1}$$

Then the result of the transformation of (3.2) together with arbitrary higher order terms, and possibly other irrelevant terms in the normal form, takes again the same form, except that all the higher order terms and the other irrelevant terms are put into terms of order at least ε. Therefore taking the limit of $\varepsilon \to 0$, we have the limit system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \bar{\alpha}x - x^3 + \bar{A}y + \bar{B}xz + \bar{C}yz, \\ \dot{z} &= \bar{\gamma}z + x^2, \end{aligned} \tag{5.2}$$

which is a 5-parameter family of polynomial vector fields of degree 3. Since we have made a rescaling of parameters as well, in general the parameters $\bar{\alpha}, \bar{\gamma}, \bar{A}, \bar{B}, \bar{C}$ are no longer small. It is known by [Guckenheimer and Williams

1979] and [Robinson 1981] that the geometric Lorenz attractor is persistent under C^r-perturbation for large enough r, although it is structurally unstable. Therefore, if one has a geometric Lorenz attractor in (5.2) with certain parameter values, then it keeps existing in nearby vector fields with $\varepsilon \neq 0$, and hence in an unfolding of a Lorenz singularity. In what follows we assume that

$$\bar{\alpha} > 0, \quad \bar{\gamma} < 0, \quad \frac{1}{2} < -\frac{\bar{\gamma}}{\sqrt{\bar{\alpha}}} < 1$$

and $\bar{A}, \bar{B}, \bar{C}$ are still small, so that we regard the equation (5.2) as a 3-parameter perturbation of the vector field:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \bar{\alpha}x - x^3, \\ \dot{z} &= \bar{\gamma}z + x^2. \end{aligned} \tag{5.3}$$

This equation plays a crucial role throughout this paper. For the sake of simplicity of the notation, we will suppress the bars over the parameters in the sequel.

6 Inclination-flip homoclinic orbits

Consider a vector field v on $\mathbb{R}^3$ and suppose the vector field admits a homoclinic orbit Γ to a hyperbolic equilibrium point O. Here we assume that the linearization matrix $Dv(O)$ at the equilibrium point has three real eigenvalues $\lambda^u, \lambda^s, \lambda^{ss}$ with

$$\lambda^{ss} < \lambda^s < 0 < \lambda^u.$$

This implies the dimension of the unstable and stable manifolds of O being

$$\dim W^u(O) = 1 \quad \text{and} \quad \dim W^s(O) = 2,$$

and, in particular, one of the branches of the one-dimensional unstable manifold is nothing but the homoclinic orbit Γ.

In order to give the definition of the inclination-flip homoclinic orbit, we first consider an invariant manifold which is tangent to the eigendirections associated with λ^u and λ^s. Such an invariant manifold exists due to the general theory from [Hirsch, Pugh and Shub 1977], but it is not necessarily unique. In this paper we call this manifold an *extended unstable manifold* and denote it by $W^{eu}(O)$. By definition, the extended unstable manifold contains the unstable manifold, and hence we can prolong it, by the time-reversed flow, along the homoclinic orbit Γ.

Definition 6.1. The homoclinic orbit Γ of the vector field v is called *inclination-flip*, if the two invariant manifolds $W^s(O)$ and $W^{eu}(O)$ are tangent along Γ, and moreover, as non-degeneracy conditions, the following is satisfied:

(Ev): $\lambda^u \neq |\lambda^s|$;

(Asy): Γ is tangent at O to the eigendirection associated to λ^s.

Note that, in spite of the non-uniqueness of the extended unstable manifolds, their tangent space along the homoclinic orbit is uniquely determined, and hence the above definition is well-defined.

For a generic homoclinic orbit, one can define its twistedness. Namely in our context, a homoclinic orbit is said to be *twisted* [resp. *non-twisted*] if the stable manifold forms a Möbius band [resp. a cylinder] in a tubular neighborhood of the homoclinic orbit. The inclination-flip homoclinic orbit is then a degenerate situation in such a way that it lies in transition between twisted and non-twisted homoclinic orbits (Figure 6.1). For a historical note, see the following Remark.

Let v be a g-equivariant vector field on $\mathbb{R}^3$ satisfying (GL1) and, instead of (GL2,3), we assume that it admits an inclination-flip homoclinic orbit Γ_+ to the origin O. The symmetry then implies that there exists another inclination-flip homoclinic orbit $\Gamma_- = g(\Gamma_+)$. We call such a pair $\Gamma_\pm$ an *inclination-flip double homoclinic loop.* The following remarkable theorem is due to [Rychlik 1990].

Theorem 6.2. (Rychlik) *Let v be a g-equivariant vector field with an inclination-flip double homoclinic loop to a hyperbolic equilibrium point whose eigenvalues $\lambda^{ss} < \lambda^s < 0 < \lambda^u$ satisfy*

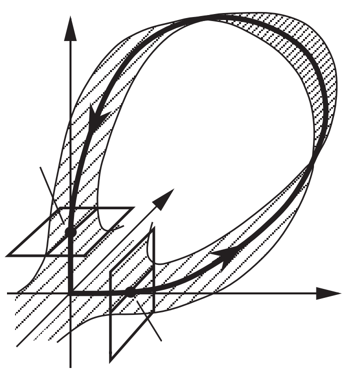
$$\frac{1}{2} < -\frac{\lambda^s}{\lambda^u} < 1 < -\frac{\lambda^{ss}}{\lambda^u}, \tag{6.1}$$

and let v_μ be its generic unfolding with $v_0 = v$. Then there exists an arbitrarily small μ such that v_μ possesses a geometric Lorenz attractor.

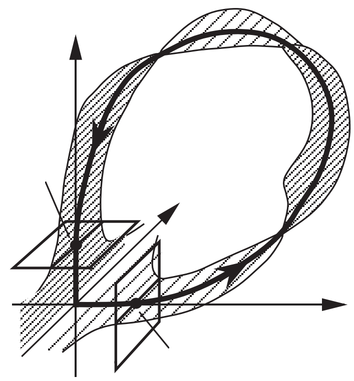
Here we can take μ as a two-dimensional parameter.

Remark 6.3. The inclination-flip homoclinic orbit was, to the authors' knowledge, first introduced by [Yanagida 1987] for the study of homoclinic doubling bifurcations, although he did not use the terminology "inclination-flip" suggested by B. Deng later. In fact Yanagida introduced three types of codimension two degenerate homoclinic orbits which are now called

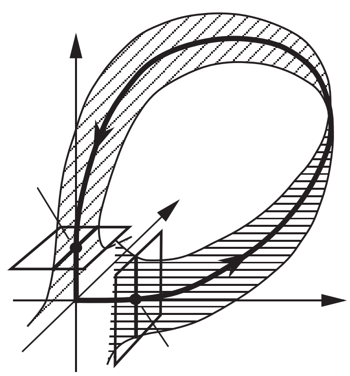
1. a homoclinic orbit with resonance;



(a) Non-twisted



(b) Twisted



(c) Critically twisted with respect to the unstable manifold

Figure 6.1: An inclination-flip homoclinic orbit.

2. an inclination-flip homoclinic orbit;

3. an orbit-flip homoclinic orbit,

each of which undergoes a homoclinic doubling bifurcation ([Chow-Deng-Fiedler 1990], [Kisaka-Kokubu-Oka 1993], [Sandstede 1993]). Such homoclinic doubling bifurcations turn out to be closely related to a homoclinic bifurcation giving rise to geometric Lorenz attractors. Indeed, the above theorem due to Rychlik shows the creation of geometric Lorenz attractors from an inclination-flip double homoclinic loop, and, following Rychlik's work, Robinson proved the birth of geometric Lorenz attractors from a double homoclinic loop with resonance ([Robinson 1989, 1992]). More recently, it is shown an analogous result for an orbit-flip double homoclinic loop ([Kokubu and Oka]), which completes the similarity of homoclinic doubling bifurcations and homoclinic bifurcations generating geometric Lorenz attractors. In this paper we use the Rychlik's theorem because it is easier to construct an inclination-flip homoclinic loop in an unfolding of a degenerate singularity. See also [Oka 1994].

The following is the fundamental observation given by [Rychlik 1990]: The equation (5.3) admits a symmetric pair of inclination-flip homoclinic orbits. Indeed, the first two equations are independent on the z-variable and are integrable due to the Hamiltonian structure. Thus we can obtain the explicit double homoclinic solution $(x(t), y(t))$ to the hyperbolic equilibrium point $(0,0)$ where the linearization has the eigenvalues $\pm\sqrt{\alpha}$. Then we substitute the known function $x(t)$ into the third equation of (5.3) and solve it using the variation of constants formula yielding

$$z(t) = e^{\gamma t}\left(z(0) + \int_0^t e^{-\gamma s} x(s)^2 ds\right).$$

Take $z(0) = -\int_0^{-\infty} e^{-\gamma s} x(s)^2 ds$, namely, we take

$$z(t) = e^{\gamma t}\int_{-\infty}^t e^{-\gamma s} x(s)^2 ds. \tag{6.2}$$

We claim that this solution $h(t) = (x(t), y(t), z(t))$ is an inclination-flip homoclinic orbit to the equilibrium point $O = (0,0,0)$. In fact, since the eigenvalues at O is given by $\lambda^u = \sqrt{\alpha}, \lambda^s = \gamma, \lambda^{ss} = -\sqrt{\alpha}$ from the assumption $0 < -\gamma < \sqrt{\alpha}$, the condition (Ev) is satisfied. Then the orbit $h(t)$ is indeed homoclinic to O, since $x(t)$ has the following asymptotic behavior:

$$|x(t)| = O(e^{\sqrt{\alpha}t}) \text{ as } t \to -\infty \quad \text{and} \quad |x(t)| = O(e^{-\sqrt{\alpha}t}) \text{ as } t \to +\infty,$$

from which it is easy to show that $z(t) \to 0$ as $t \to \pm\infty$. The same estimate in fact shows that $|z(t)| = O(e^{\gamma t})$ as $t \to +\infty$, which verifies the condition (Asy). Finally in order to check the inclination-flip condition, it suffices to see that the surface given by the direct product of the homoclinic orbit $(x(t), y(t))$ in the (x, y)-plane with the z-axis is invariant under the flow. This implies that the surface is nothing but the stable manifold and at the same time the extended unstable manifold. However, the system (5.3) does not satisfy the condition (6.1) in the Rychlik's theorem, since $\lambda^u = |\lambda^{ss}| = \sqrt{\alpha}$, and hence we cannot directly apply the theorem to (5.3). Therefore we need to add more parameters to perturb the equation so that it recovers the desired eigenvalue condition $\lambda^u < |\lambda^{ss}|$ without breaking the inclination-flip double homoclinic loop. In the next section we derive a persistence condition for inclination-flip homoclinic orbits, which plays an essential role in our perturbation argument.

7 Persistence of inclination-flip homoclinic orbits

Let $\dot{\mathbf{x}} = v(\mathbf{x}, \mu)$ be a k-parameter family ($k \geq 3$) of vector fields on $\mathbb{R}^3$ such that, for $\mu = 0$, the vector field $\dot{\mathbf{x}} = v(\mathbf{x}, 0)$ possesses an inclination-flip homoclinic orbit $h(t)$ to a hyperbolic equilibrium point O. Namely the extended unstable manifold $W^{eu}(O)$ is tangent to the stable manifold $W^s(O)$ along the homoclinic orbit $h(t)$. Let $\lambda^s, \lambda^{ss}, \lambda^u$ be corresponding eigenvalues at O. In this section, we shall study the condition for the persistence of the inclination-flip homoclinic orbit under the perturbation by μ.

Consider the variational equation along $h(t)$ for $\mu = 0$:

$$\dot{u} = Dv(h(t), 0)u$$

and take the three linearly independent solutions $q_0(t) = \dot{h}(t), q_1(t), q_2(t)$ to the variational equations satisfying the following asymptotic behavior:

$$\begin{array}{llllll} |q_0(t)| & = O(e^{\lambda^u t}) & \text{as } t \to -\infty & \text{and} & |q_0(t)| & = O(e^{\lambda^s t}) \quad \text{as } t \to +\infty, \\ |q_1(t)| & = O(e^{\lambda^s t}) & \text{as } t \to -\infty & \text{and} & |q_1(t)| & = O(e^{\lambda^{ss} t}) \quad \text{as } t \to +\infty, \\ |q_2(t)| & = O(e^{\lambda^{ss} t}) & \text{as } t \to -\infty & \text{and} & |q_2(t)| & = O(e^{\lambda^u t}) \quad \text{as } t \to +\infty. \end{array}$$

For the proof of the existence of such solutions, see [Gruendler 1985] and [Kokubu 1988]. Note that these fundamental solutions span tangent spaces of the invariant manifolds along the inclination-flip homoclinic orbit as follows:

$$T_{h(t)}W^s(O) = T_{h(t)}W^{eu}(O) = \text{span}\{q_0(t), q_1(t)\}, \quad T_{h(t)}W^u(O) = \text{span}\{q_0(t)\}.$$

We regard these solutions as column vector functions and define the fundamental matrix solution to the variational equation by

$$V(t) = (q_0(t), q_1(t), q_2(t)).$$

Also we take the projection matrix

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and define the column vector function

$$X(t) = V(t)\left[P\int_{-\infty}^{t} V(s)^{-1}\frac{\partial v}{\partial \mu}(h(s),0)ds + (I-P)\int_{0}^{t} V(s)^{-1}\frac{\partial v}{\partial \mu}(h(s),0)ds\right].$$

Theorem 7.1. *The inclination-flip homoclinic orbit persists along a codimension two submanifold $\mathcal{B}$ in the μ-space, provided the following two vectors are linearly independent:*

$$\begin{aligned} M_1 = & \int_{-\infty}^{\infty} \hat{b}(t)\tfrac{\partial v}{\partial \mu}(h(t),0)dt, \\ M_2 = & \int_{-\infty}^{\infty} \hat{b}(t)\left\{D^2 v(h(t),0)X(t) + \tfrac{\partial}{\partial \mu}Dv(h(t),0)\right\} q_1(t)dt, \end{aligned}$$

where $\hat{b}(t)$ is a unique (up to constant multiple) bounded solution to the adjoint variational equation

$$\dot{\hat{u}} = -\hat{u}\cdot Dv(h(t),0)$$

along the homoclinic orbit $h(t)$.

Furthermore the submanifold $\mathcal{B}$ is perpendicular to M_1 and M_2 at $\mu = 0$.

Sometimes, it is more convenient to take advantage of special form of equations, namely, the following Corollary holds:

Corollary 7.2. *Suppose the unperturbed system $\dot{\mathbf{x}} = v(\mathbf{x}, 0)$ at $\mu = 0$ takes the form*

$$\begin{aligned} \dot{x} = & \ v_1(x,y), \\ \dot{y} = & \ v_2(x,y), \\ \dot{z} = & \ \gamma z + v_3(x,y), \end{aligned}$$

where the $(\dot{x}, \dot{y})$-equation is a Hamiltonian equation with a homoclinic orbit to a hyperbolic saddle point, say O, and $\gamma < 0$ is a weaker stable eigenvalue at O so that the corresponding homoclinic orbit in the entire vector field becomes an inclination-flip one. Then the second integral M_2 for the persistence of the inclination-flip homoclinic orbit is simplified to

$$M_2 = \int_{-\infty}^{\infty} \hat{b}(t) \frac{\partial}{\partial \mu} Dv(h(t), 0) e(t) dt,$$

where $e(t) = (0, 0, e^{\gamma t})^T$ *and* $\hat{b}(t) = (\dot{y}(t), -\dot{x}(t), 0)$.

Proof of Theorem 7.1. The proof mainly follows the idea of Theorem A in [Kokubu 1988]. Indeed the derivation of the integral M_1 has been done there, however we give an outline of it here since it is used for the derivation of the other integral M_2.

Take the inverse of the fundamental matrix

$$V(t) = (q_0(t), q_1(t), q_2(t)),$$

then it gives a fundamental matrix for the adjoint variational equation

$$\dot{\hat{u}} = -\hat{u} \cdot Dv(h(t), 0), \tag{7.1}$$

where $\hat{u}$ stands for a three-dimensional row vector. Let the row vector functions $\hat{q}_i(t)$ $(i = 0, 1, 2)$ be defined by

$$V(t)^{-1} = \begin{pmatrix} \hat{q}_0(t) \\ \hat{q}_1(t) \\ \hat{q}_2(t) \end{pmatrix},$$

which are fundamental solutions to (7.1). In particular, the solution $\hat{q}_2(t)$ is a unique (up to constant multiple) non-trivial bounded solution.

The main idea of the proof of Theorem 7.1 is to take a cross section

$$\Sigma = \text{span}\{q_1(t), q_2(t)\}$$

transverse to the homoclinic orbit $h(t)$ and to draw perturbed invariant manifolds $W^u(O;\mu), W^s(O;\mu), W^{eu}(O;\mu)$ on this section Σ. For this purpose, it is convenient to make the change of variable

$$x = h(t) + z,$$

and to rewrite the original equation as

$$\dot{z} = Dv(h(t), 0)z + N(t, z, \mu), \tag{7.2}$$

where

$$N(t, z, \mu) = v(h(t) + z, \mu) - v(h(t), 0) - Dv(h(t), 0)z. \tag{7.3}$$

Furthermore, take the initial condition of z as

$$z(0) = \xi_1 q_1(0) + \xi_2 q_2(0) \tag{7.4}$$

and denote the solution with such an initial condition by

$$z(t; \xi, \mu), \qquad \xi = (\xi_1, \xi_2).$$

If there is no confusion, we sometimes identify the initial point $z(0; \xi, \mu)$ with ξ itself. Since $V(t)$ is a fundamental matrix, the variation of constants formula convert the equation (7.2) to the following equivalent integral equation:

$$z(t) = V(t) \left\{ V(0)^{-1} z(0) + \int_0^t V(s)^{-1} N(s, z(s), \mu) ds \right\}. \tag{7.5}$$

The next lemma immediately follows from the definition of the fundamental matrix $V(t)$ and the asymptotic behavior of the fundamental solutions $q_i(t)$ $(i = 0, 1, 2)$.

Lemma 7.3. (exponential dichotomy)
(i) Let P^- be the projection matrix given by

$$P^- = \mathrm{diag}(0, 1, 1),$$

then there exists positive constants K, α such that

$$\begin{aligned} |V(t)(I - P^-)V(s)^{-1}| &\leq Ke^{-\alpha(s-t)}, \quad (t \leq s \leq 0), \\ |V(t)P^- V(s)^{-1}| &\leq Ke^{-\alpha(t-s)}, \quad (s \leq t \leq 0); \end{aligned}$$

(ii) Let P^+ be the projection matrix given by

$$P^+ = \mathrm{diag}(1, 1, 0),$$

then there exists positive constants K, α such that

$$\begin{aligned} |V(t)P^+ V(s)^{-1}| &\leq Ke^{-\alpha(t-s)}, \quad (0 \leq s \leq t), \\ |V(t)(I - P^+)V(s)^{-1}| &\leq Ke^{-\alpha(s-t)}, \quad (0 \leq t \leq s). \end{aligned}$$

Note that $I - P^-$ is the projection to $T_{h(0)}W^u(O)$ and P^+ is the projection to $T_{h(0)}W^s(O)$.

For $t \leq 0$, we decompose (7.5) into

$$\begin{aligned} z(t) &= V(t)(I - P^-) \left\{ V(0)^{-1} z(0) + \int_0^t V(s)^{-1} N(s, z(s), \mu) ds \right\} \\ &\quad + V(t)P^- \left\{ V(0)^{-1} z(0) + \int_0^t V(s)^{-1} N(s, z(s), \mu) ds \right\}. \end{aligned}$$

From the estimate in the previous lemma, we can show that the first term of the integral equation stays bounded as $t \to -\infty$ whereas the second term diverges to ∞ unless

$$P^- \left\{ V(0)^{-1} z(0) + \int_0^{-\infty} V(s)^{-1} N(s, z(s), \mu) ds \right\} = 0. \qquad (7.6)$$

Denote the left hand side by $E^-(\xi, \mu)$, then the above condition shows in fact that $E^-(\xi, \mu) = 0$ if and only if $\xi \in W^u(O; \mu)$. By the implicit function theorem, we can solve the equation $E^-(\xi, \mu) = 0$ as:

$$\xi = \xi^-(\mu) = (\xi_1^-(\mu), \xi_2^-(\mu))$$

which gives a point of intersection in Σ with the unstable manifold $W^u(O; \mu)$.

Similarly for $t \geq 0$, we obtain the condition of the stable manifold $W^s(O; \mu)$ as:

$$E^+(\xi, \mu) = (I - P^+) \left\{ V(0)^{-1} z(0) + \int_0^{\infty} V(s)^{-1} N(s, z(s), \mu) ds \right\} = 0,$$

which has a solution of the form:

$$\xi_2 = \xi_2^+(\xi_1, \mu)$$

corresponding to the intersection curve of the stable manifold $W^s(O; \mu)$ with the section Σ.

Now the set of persistence for a homoclinic orbit is given by

$$\mathcal{H} = \{\mu \mid \xi_2^-(\mu) - \xi_2^+(\xi_1^-(\mu), \mu) = 0\}$$

and its gradient vector at $\mu = 0$ is indeed

$$M_1 = \int_{-\infty}^{\infty} \hat{b}(t) \frac{\partial v}{\partial \mu}(h(t), 0) dt.$$

This proves the first half of Theorem 7.1. For more detail, see [Kokubu 1988].

For the persistence of the inclination-flip condition, let us first take functions given by

$$h^\pm(t, \mu) = h(t) + z(t; \xi^\pm(\mu), \mu)$$

where

$$\begin{aligned} \xi^-(\mu) &= (\xi_1^-(\mu), \xi_2^-(\mu)) \\ \xi^+(\mu) &= (\xi_1^-(\mu), \xi_2^+(\xi_1^-(\mu), \mu)). \end{aligned}$$

These functions $h^\pm(t; \mu)$ converge to the equilibrium point exponentially as $t \to \pm\infty$, respectively, since $h^-(0; \mu) \in W^u(O; \mu)$ and $h^+(0; \mu) \in W^s(O; \mu)$.

Consider the variational equation along these half orbits:

$$\begin{aligned} \dot{u} &= Dv(h^{\pm}(t;\mu),\mu)u \\ &= \{Dv(h(t),0) + R^{\pm}(t,\mu)\}u, \end{aligned} \tag{7.7}$$

where

$$R^{\pm}(t,\mu) = Dv(h^{\pm}(t;\mu),\mu) - Dv(h(t),0).$$

Taking the initial condition

$$u(0) = \zeta_1 q_1(0) + \zeta_2 q_2(0),$$

we denote the solution to (7.7) by $u^{\pm}(t;\zeta,\mu)$. Then the similar argument as before yields that $\zeta = (\zeta_1,\zeta_2) \in T_{h^+(0,\mu)}W^s(O;\mu)$ if and only if

$$(I - P^+)\left\{V(0)^{-1}u^+(0;\zeta,\mu) + \int_0^{+\infty} V(s)^{-1}R^+(s,\mu)u^+(s;\zeta,\mu)ds\right\} = 0.$$

Since $u^+(s;\zeta,\mu)$ is linear in ζ, the latter condition takes the form

$$K^+(\mu)\zeta_1 + (1 + L^+(\mu))\zeta_2 = 0. \tag{7.8}$$

On the other hand, for $T_{h^-(0,\mu)}W^{eu}(O;\mu)$, it is more convenient to consider

$$w^-(t;\zeta,\mu) = e^{-\lambda t}u^-(t;\zeta,\mu),$$

where λ is a real number satisfying $\lambda^{ss} < \lambda < \lambda^s < 0$. Clearly $\zeta \in T_{h^-(0,\mu)}W^{eu}(O;\mu)$ if and only if $w^-(t;\zeta,\mu) \to 0$ as $t \to -\infty$, or equivalently $|w^-(t;\zeta,\mu)|$ remains bounded as $t \to -\infty$.

The equation (7.7) with minus-sign then takes

$$\begin{aligned} \dot{w}^-(t) &= -\lambda e^{-\lambda t}u^-(t) + e^{-\lambda t}\dot{u}^-(t) \\ &= -\lambda w^-(t) + \{Dv(h(t),0) + R^-(t,\mu)\}w^-(t) \\ &= \{(Dv(h(t),0) - \lambda I) + R^-(t,\mu)\}w^-(t), \end{aligned}$$

which has a fundamental matrix

$$W(t) = e^{-\lambda t}V(t)$$

when $\mu = 0$, since $R^{\pm}(t,0) = 0$. In particular, we have the following exponential dichotomy estimate:

$$\begin{aligned} |W(t)(I - Q^-)W(s)^{-1}| &\leq Ke^{-\alpha(s-t)}, \quad (t \leq s \leq 0), \\ |W(t)Q^-W(s)^{-1}| &\leq Ke^{-\alpha(t-s)}, \quad (s \leq t \leq 0), \end{aligned}$$

for some $K, \alpha > 0$, where $Q^- = \text{diag}(0,0,1)$, that is $I - Q^-$ is the projection to $T_{h^-(0,\mu)}W^{eu}(O;\mu)$ which is spanned by $q_0(0)$ and $q_1(0)$.

Now the same argument works for this case as well and we see that $\zeta = (\zeta_1, \zeta_2) \in T_{h^-(0,\mu)}W^{eu}(O;\mu)$ if and only if

$$\begin{aligned} &Q^- \left\{ W(0)^{-1}w^-(0;\zeta,\mu) - \int_{-\infty}^0 W(s)^{-1}R^-(s,\mu)w^-(s;\zeta,\mu)ds \right\} \\ &= Q^- \left\{ V(0)^{-1}u^-(0;\zeta,\mu) - \int_{-\infty}^0 V(s)^{-1}R^-(s,\mu)u^-(s;\zeta,\mu)ds \right\} = 0. \end{aligned}$$

From the linearity of $u^-(s;\zeta,\mu)$ with respect to ζ, the last condition takes the form

$$K^-(\mu)\zeta_1 + (1 + L^-(\mu))\zeta_2 = 0. \tag{7.9}$$

Note that $Q^- = I - P^+$.

The condition for the inclination-flip is therefore given by the equation

$$-\frac{K^-(\mu)}{1 + L^-(\mu)} = -\frac{K^+(\mu)}{1 + L^+(\mu)} \tag{7.10}$$

which defines a set $\mathcal{T}$ in the parameter space. Then the set

$$\mathcal{C} = \mathcal{H} \cap \mathcal{T}$$

gives the desired set of parameters for which the original equation $\dot{\mathbf{x}} = v(\mathbf{x}, \mu)$ has an inclination-flip homoclinic orbit. Our remaining task is to show that the set $\mathcal{T}$ defines a local submanifold of codimension one whose gradient vector at $\mu = 0$ is spanned by the integrals M_1 and M_2. Then the conclusion of Theorem 7.1 immediately follows from the implicit function theorem.

From the defining equation (7.10) of $\mathcal{T}$, its gradient vector at $\mu = 0$ is given by

$$-(K^-)'(0) + (K^+)'(0).$$

We shall compute the derivatives $(K^\pm)'(0)$ as follows: First we have

$$\begin{aligned} &\left.\frac{\partial}{\partial\mu}\right|_{\mu=0} Q^- \left\{ V(0)^{-1}u^-(0;\zeta,\mu) - \int_{-\infty}^0 V(s)^{-1}R^-(s,\mu)u^-(s;\zeta,\mu)ds \right\} \\ &= -Q^- \int_{-\infty}^0 V(s)^{-1}\frac{\partial R^-}{\partial\mu}(s,0)u^-(s;\zeta,0)ds \\ &\qquad -Q^- \int_{-\infty}^0 V(s)^{-1}R^-(s,0)\frac{\partial u^-}{\partial\mu}(s;\zeta,0)ds. \end{aligned}$$

Then from the definition of the initial condition

$$u^-(0;\zeta,\mu) = \zeta_1 q_1(0) + \zeta_2 q_2(0)$$

and from

$$\begin{aligned} R^-(s,0) &\equiv 0, \\ \frac{\partial R^-}{\partial \mu}(s,0) &= D^2 v(h(s),0)\frac{\partial h^-}{\partial \mu}(s,0) + \frac{\partial}{\partial \mu} Dv(h(s),0), \end{aligned}$$

we have

$$(K^-)'(0) = \int_{-\infty}^{0} \hat{q}_2(s)\left\{ D^2 v(h(s),0)\frac{\partial h^-}{\partial \mu}(s,0) + \frac{\partial}{\partial \mu} Dv(h(s),0)\right\} q_1(s)ds.$$

Similarly, we have

$$(K^+)'(0) = -\int_{0}^{+\infty} \hat{q}_2(s)\left\{ D^2 v(h(s),0)\frac{\partial h^+}{\partial \mu}(s,0) + \frac{\partial}{\partial \mu} Dv(h(s),0)\right\} q_1(s)ds.$$

Lemma 7.4.
(i) The function $\frac{\partial h^-}{\partial \mu}(t,0)$ *coincides with the column vector function* $X(t)$*, namely,*

$$\begin{aligned} &\frac{\partial h^-}{\partial \mu}(t,0) \\ &\quad = V(t)\left[P\int_{-\infty}^{t} V(s)^{-1}\frac{\partial v}{\partial \mu}(h(s),0)ds + (I-P)\int_{0}^{t} V(s)^{-1}\frac{\partial v}{\partial \mu}(h(s),0)ds \right]. \end{aligned}$$

where $P = P^-$*.*
(ii)

$$\frac{\partial h^+}{\partial \mu}(t,0) - \frac{\partial h^-}{\partial \mu}(t,0) = V(t)\begin{pmatrix} 0 \\ 0 \\ M_1 \end{pmatrix}.$$

From this Lemma, the conclusion of Theorem 7.1 immediately follows, since

$$-(K^-)'(0) + (K^+)'(0) = -M_2 + (\text{const.})M_1.$$

Below we shall prove Lemma 7.4.

Proof. Recall that the function $h^{\pm}(t,\mu)$ satisfies

$$\frac{d}{dt}h^{\pm}(t,\mu) = v(h^{\pm}(t,\mu),\mu),$$

and hence $\frac{d}{dt}\frac{\partial h^{\pm}}{\partial \mu}(t,0)$ satisfies the linear inhomogeneous equation

$$\frac{d}{dt}\frac{\partial h^{\pm}}{\partial \mu}(t,0) = Dv(h(t),0)\frac{\partial h^{\pm}}{\partial \mu}(t,0) + \frac{\partial v}{\partial \mu}(h(t),0).$$

Therefore we have

$$\frac{\partial h^{\pm}}{\partial \mu}(t,0) = V(t)\left\{V(0)^{-1}\frac{\partial h^{\pm}}{\partial \mu}(0,0) + \int_0^t V(s)^{-1}\frac{\partial v}{\partial \mu}(h(t),0)\right\}, \tag{7.11}$$

which yields

$$\begin{aligned}\frac{\partial h^{+}}{\partial \mu}(t,0) - \frac{\partial h^{-}}{\partial \mu}(t,0) &= V(t)V(0)^{-1}\left(\frac{\partial h^{+}}{\partial \mu}(0,0) - \frac{\partial h^{-}}{\partial \mu}(0,0)\right)\\ &= V(t)\begin{pmatrix} 0 \\ 0 \\ M_1 \end{pmatrix}\end{aligned}$$

from the definition of $h^{\pm}(t,\mu)$. This proves the statement (ii).

We shall show the statement (i). Since $\frac{\partial h^{-}}{\partial \mu}(t,0)$ converges to 0 as $t \to -\infty$, the same argument for obtaining (7.6) applied to (7.11) yields

$$P\left\{V(0)^{-1}\frac{\partial h^{-}}{\partial \mu}(0,0) + \int_0^{-\infty} V(s)^{-1}\frac{\partial v}{\partial \mu}(h(s),0)ds\right\} = 0.$$

Here we have used that

$$V(0)^{-1}\frac{\partial h^{-}}{\partial \mu}(0,0) = \begin{pmatrix} 0 \\ {\xi_1^{-}}'(0) \\ {\xi_2^{-}}'(0) \end{pmatrix} = PV(0)^{-1}\frac{\partial h^{-}}{\partial \mu}(0,0).$$

From this together with (7.11), we obtain the desired equality. This completes the proof of Lemma 7.4 and hence of Theorem 7.1. □

Now we proceed to the proof of Corollary 7.2.

Proof of Corollary 7.2. The special form of the equation implies that we can choose

$$q_1(t) = \dot{h}(t) + c \cdot e(t)$$

where c is a constant.

Lemma 7.5.

$$\begin{aligned}&\hat{q}_2(t)\left\{D^2v(h(t),0)\frac{\partial h^{\pm}}{\partial \mu}(t,0) + \frac{\partial}{\partial \mu}Dv(h(t),0)\right\}\dot{h}(t)\\ &= \frac{d}{dt}\left\{\hat{q}_2(t)\cdot\frac{d}{dt}\left(\frac{\partial h^{\pm}}{\partial \mu}(t,0)\right)\right\}.\end{aligned}$$

Proof. Straightforward computation using

$$\frac{d}{dt}\hat{q}_2(t) = -\hat{q}_2(t)Dv(h(t),0)$$

and

$$\frac{d}{dt}\frac{\partial h^{\pm}}{\partial \mu}(t,0) = Dv(h(t),0)\frac{\partial h^{\pm}}{\partial \mu}(t,0) + \frac{\partial v}{\partial \mu}(h(t),0)$$

prove the desired equality. □

From this lemma with $q_1(t) = \dot{h}(t) + c \cdot e(t)$, we have

$$\begin{aligned}
&\int_0^{+\infty} \hat{q}_2(s)\left\{D^2v(h(s),0)\frac{\partial h^+}{\partial \mu}(s,0) + \frac{\partial}{\partial \mu}Dv(h(s),0)\right\} q_1(s)ds \\
&\quad = \int_0^{+\infty} \frac{d}{dt}\left\{\hat{q}_2(t)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(t,0)\right)\right\} dt \\
&\quad\quad + c\int_0^{+\infty} \hat{q}_2(s)\left\{D^2v(h(s),0)\frac{\partial h^+}{\partial \mu}(s,0) + \frac{\partial}{\partial \mu}Dv(h(s),0)\right\} e(t)ds.
\end{aligned} \tag{7.12}$$

The first term can be written as

$$\begin{aligned}
&\int_0^{+\infty} \frac{d}{dt}\left\{\hat{q}_2(t)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(t,0)\right)\right\} dt \\
&\quad = \left[\hat{q}_2(t)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(t,0)\right)\right]_0^{+\infty} \\
&\quad = \lim_{t\to+\infty}\hat{q}_2(t)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(t,0)\right) - \hat{q}_2(0)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(0,0)\right) \\
&\quad = -\hat{q}_2(0)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(0,0)\right),
\end{aligned}$$

since, as $t \to +\infty$, $\hat{q}_2(t)$ and $\frac{d}{dt}\left(\frac{\partial h^+}{\partial \mu}(t,0)\right)$ both converge to 0 exponentially.

From the special form of the vector field, we have

$$D^2v(h(t),0)e(t) \equiv 0,$$

and hence, for the second term of (7.12),

$$\begin{aligned}
&\int_0^{+\infty} \hat{q}_2(t)\left\{D^2v(h(t),0)\frac{\partial h^+}{\partial \mu}(t,0) + \frac{\partial}{\partial \mu}Dv(h(t),0)\right\} e(t)dt \\
&\quad = \int_0^{+\infty} \hat{q}_2(t)\frac{\partial}{\partial \mu}Dv(h(t),0)e(t)dt.
\end{aligned}$$

Thus we have obtained

$$
\begin{aligned}
(K^+)'(0) = \; & -c\int_0^{+\infty} \hat{q}_2(t)\frac{\partial}{\partial\mu}Dv(h(t),0)e(t)dt \\
& +\hat{q}_2(0)\cdot\frac{d}{dt}\left(\frac{\partial h^+}{\partial\mu}(0,0)\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(K^-)'(0) = c\int_{-\infty}^{0} \hat{q}_2(t)\frac{\partial}{\partial\mu}Dv(h(t),0)e(t)dt \\
+\hat{q}_2(0)\cdot\frac{d}{dt}\left(\frac{\partial h^-}{\partial\mu}(0,0)\right).
\end{aligned}
$$

Therefore the gradient vector to the set $\mathcal{T}$ at $\mu = 0$ is given by

$$
\begin{aligned}
-(K^-)'(0) + (K^+)'(0) = -c\int_{-\infty}^{+\infty} \hat{q}_2(t)\frac{\partial}{\partial\mu}Dv(h(t),0)e(t)dt \\
+\hat{q}_2(0)\cdot\left\{\frac{d}{dt}\left(\frac{\partial h^+}{\partial\mu}(0,0)\right) - \frac{d}{dt}\left(\frac{\partial h^-}{\partial\mu}(0,0)\right)\right\},
\end{aligned}
$$

the first term of which is nothing but the desired simplified form of M_2, since, for the special form of vector field, it is easy to see that bounded fundamental solution $\hat{q}_2(t)$ is given by

$$
\hat{q}_2(t) = \hat{b}(t) = (\dot{y}(t), -\dot{x}(t), 0).
$$

Finally we compute

$$
\hat{q}_2(0)\cdot\left\{\frac{d}{dt}\left(\frac{\partial h^+}{\partial\mu}(0,0)\right) - \frac{d}{dt}\left(\frac{\partial h^-}{\partial\mu}(0,0)\right)\right\}.
$$

From

$$
\frac{d}{dt}\frac{\partial h^\pm}{\partial\mu}(t,0) = Dv(h(t),0)\frac{\partial h^\pm}{\partial\mu}(t,0) + \frac{\partial v}{\partial\mu}(h(t),0),
$$

we have

$$
\frac{d}{dt}\frac{\partial h^+}{\partial\mu}(0,0) - \frac{d}{dt}\frac{\partial h^-}{\partial\mu}(0,0) = Dv(h(0),0)V(0)\begin{pmatrix} 0 \\ 0 \\ M_1 \end{pmatrix},
$$

and hence

$$
\hat{q}_2(0)\cdot\left\{\frac{d}{dt}\left(\frac{\partial h^+}{\partial\mu}(0,0)\right) - \frac{d}{dt}\left(\frac{\partial h^-}{\partial\mu}(0,0)\right)\right\} = (\text{const.})M_1.
$$

This completes the proof of Corollary 7.2. $\square$

8 Computation of the integrals

Take the vector field:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \alpha x - x^3 + Ay + Bxz + Cyz,\\ \dot{z} &= \gamma z + x^2,\end{aligned}$$

where $\mu = (A, B, C)$ as a perturbation of (5.3) which has an inclination-flip homoclinic orbit.

It is not hard to give all the necessary information explicitly for the computation of the integrals M_1 and M_2 using the original formula in Theorem 7.1 for this case. Indeed, since the homoclinic solution $h(t)$ is given, it is easy to obtain $q_0(t) = \dot{h}(t)$. The solution $q_1(t)$ is given of the form $q_1(t) = q_0(t) + ce(t)$ where $e(t) = (0, 0, e^{\gamma t})^T$ and the constant c is chosen so as to satisfy $|q_1(t)| = O(e^{\lambda^s t})$ as $t \to -\infty$. For $q_2(t)$, we first consider the Hamiltonian function $H(x, y) = \frac{y^2}{2} - \frac{\alpha x^2}{2} + \frac{x^4}{4}$ for the $(\dot{x}, \dot{y})$-equation with $\mu = 0$, and note that the homoclinic orbit $h(t)$ corresponds to the energy level $H = 0$. For any $-\frac{\alpha^2}{2} < h < 0$, the energy level curve $H(x, y) = h$ gives a periodic solution $p(t; h)$. Differentiate it by h and put $h = 0$, then we have a solution $\frac{\partial}{\partial h} p(t; 0)$ to the variational equation along the homoclinic orbit for the $(\dot{x}, \dot{y})$-equation. For the $\dot{z}$-equation, again we can use the variation of constants formula and hence we obtain the desired solution $q_2(t)$. The unique bounded solution $\hat{b}(t)$ to the adjoint equation is simply given as $\hat{b}(t) = (\dot{y}(t), -\dot{x}(t), 0)$.

Therefore we can carry out the computation of M_1 and M_2 using these data. However, it is more convenient to take advantage of the special form of the equation and apply the simplified integrals given in Corollary 7.2.

Now the computation becomes much easier. In fact,

$$\begin{aligned} M_1 &= \int_{-\infty}^{\infty} (\dot{y}(t), -\dot{x}(t), 0) \begin{pmatrix} 0 & 0 & 0 \\ y(t) & x(t)z(t) & y(t)z(t) \\ 0 & 0 & 0 \end{pmatrix} dt \\ &= \int_{-\infty}^{\infty} (-\dot{x}(t)y(t), -\dot{x}(t)x(t)z(t), -\dot{x}(t)y(t)z(t))dt; \end{aligned}$$

$$\begin{aligned} M_2 &= \int_{-\infty}^{\infty} (\dot{y}(t), -\dot{x}(t), 0) \times \\ &\quad \begin{pmatrix} 0 & 0 & 0 & & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & ; & z(t) & 0 & x(t) & ; & 0 & z(t) & y(t) \\ 0 & 0 & 0 & & 0 & 0 & 0 & & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ e^{\gamma t} \end{pmatrix} dt \\ &= \int_{-\infty}^{\infty} (0, -\dot{x}(t)x(t)e^{\gamma t}, -\dot{x}(t)y(t)e^{\gamma t})dt. \end{aligned}$$

Clearly the first component of the integral M_1 and the third component of M_2 are both negative since $\dot{x}(t) = y(t)$, and therefore they are linearly independent vectors. From Theorem 7.1, we conclude that the inclination-flip double homoclinic loop persists along a one-dimensional curve in the three-dimensional parameter space (A, B, C) whose direction vector d is perpendicular to the vectors M_1 and M_2. The direction vector is hence given by

$$d = (m_B n_C - m_C n_B, -m_A n_C, m_A n_C) \tag{8.1}$$

where

$$M_1 = (m_A, m_B, m_C), \quad M_2 = (0, n_B, n_C).$$

9 Completion of proof of the main result

Finally we need to show:

Lemma 9.1. *The direction vector d has non-zero first component, namely,*

$$m_B n_C - m_C n_B \neq 0.$$

By this lemma we can make the parameter A to be negative along the persistence curve of inclination-flip homoclinic loop, and hence the linear part of the equation (5.2) at the origin verifies the desired condition of the eigenvalues

$$\frac{1}{2} < -\frac{\lambda^s}{\lambda^u} < 1 < -\frac{\lambda^{ss}}{\lambda^u}.$$

Since the other two parameters B and C unfold the inclination-flip homoclinic loop, we can now apply the Rychlik's theorem and obtain geometric Lorenz attractors in appropriately perturbed vector fields. This completes the proof of the main Theorem 3.1.

Proof of Lemma 9.1. From the computation of the integrals, we have

$$\begin{aligned} m_B &= -\int_{-\infty}^{\infty} x(t)y(t)z(t)dt, \\ m_C &= -\int_{-\infty}^{\infty} y(t)^2 z(t)dt, \\ n_B &= -\int_{-\infty}^{\infty} x(t)y(t)e^{\gamma t}dt, \\ n_C &= -\int_{-\infty}^{\infty} y(t)^2 e^{\gamma t}dt, \end{aligned}$$

where $(x(t), y(t), z(t))$ is the unperturbed inclination-flip homoclinic orbit in the equation (5.3).

Since $z(t) \geq 0$, it is easy to see $m_C < 0$ and $n_C < 0$. Notice that $x(t)$ can be taken as an even function and $y(t)$, an odd function. Then we have

$$\begin{aligned}\int_{-\infty}^{0} x(t)y(t)e^{\gamma t}dt &= \int_{\infty}^{0} x(-t)y(-t)e^{-\gamma t}(-dt) \\ &= \int_{\infty}^{0} x(t)y(t)e^{-\gamma t}dt \\ &= -\int_{0}^{\infty} x(t)y(t)e^{-\gamma t}dt,\end{aligned}$$

and hence

$$n_B = \int_0^\infty x(t)y(t)(e^{-\gamma t} - e^{\gamma t})dt$$

which is negative from $\gamma < 0$ since $e^{-\gamma t} - e^{\gamma t} \geq 0$, $x(t) \geq 0$, and $y(t) \leq 0$ for $t \geq 0$.

Similarly we have

$$m_B = \int_0^\infty x(t)y(t)\{z(-t) - z(t)\}dt.$$

We claim that $z(t) - z(-t) \geq 0$ for $t \geq 0$, which implies the desired conclusion: $m_B n_C - m_C n_B < 0$, since $m_B > 0, m_C < 0, n_B < 0$ and $n_C < 0$.

Recall from (6.2), that

$$z(t) = e^{\gamma t}\int_{-\infty}^{t} e^{-\gamma s}\text{sech}^2(-\sqrt{\alpha}s)ds,$$

and therefore, by taking $u = e^{-\gamma t} \geq 1$ for $t \geq 0$, we have

$$\begin{aligned}&z(t) - z(-t) \\ &\quad = e^{\gamma t}\int_{-\infty}^{t} e^{-\gamma s}\text{sech}^2(-\sqrt{\alpha}s)ds - e^{-\gamma t}\int_{-\infty}^{-t} e^{-\gamma s}\text{sech}^2(-\sqrt{\alpha}s)ds \\ &\quad = \tfrac{1}{u}\int_0^u \left(\frac{2}{v^c + v^{-c}}\right)^2 dv - u\int_0^{\frac{1}{u}} \left(\frac{2}{v^c + v^{-c}}\right)^2 dv\end{aligned}$$

where $c = -\sqrt{\alpha}/\gamma > 1$. The integrand

$$\varphi(v) = \left(\frac{2}{v^c + v^{-c}}\right)^2$$

is a smooth function for $v \geq 0$, monotone increasing on $[0, 1)$ and decreasing on $(1, \infty)$ and has the property:

$$\varphi(v) = \varphi(1/v).$$

For $u \geq 1$, the mean value theorem shows that there exists unique $\theta \in (0, 1/u)$ such that

$$\frac{1}{1/u}\int_0^{1/u} \varphi(v)dv = \varphi(\theta)$$

and hence

$$\begin{aligned} z(t) - z(-t) &= \tfrac{1}{u}\int_0^{u} \varphi(v)dv - \varphi(\theta) \\ &= \tfrac{1}{u}\left\{\int_0^{u} \varphi(v)dv - u \cdot \varphi(\theta)\right\} \geq 0, \end{aligned}$$

since

$$\begin{aligned} u \cdot \varphi(\theta) &= (u - 1/u)\varphi(\theta) + (1/u)\varphi(\theta) \\ &\leq \int_{1/u}^{u} \varphi(v)dv + \int_0^{1/u} \varphi(v)dv \\ &= \int_0^{u} \varphi(v)dv. \end{aligned}$$

This completes the proof of Lemma 9.1, and hence of Theorem 3.1. □

10 Related results

In [Rychlik 1990], the following equation is presented:

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = x - 2x^3 + \alpha y + \beta xz + \gamma x^2 y, \\ &\dot{z} = -\kappa z + x^2, \end{aligned}$$

as an example of a family of vector fields exhibiting geometric Lorenz attractors. Apparently, although some coefficients are different, this is essentially an extension of (5.3). It has been shown that, for $\frac{1}{2} < \kappa < 1$ with appropriate values of α and β while $\gamma = 0$, the equation satisfies the assumption of Theorem 6.2, and hence certain perturbation of it including $\gamma \neq 0$ generates geometric Lorenz attractors. The main idea here is to use a result of [Horozov 1979] and [Carr 1981], which is a symmetric version of the Bogdanov-Takens bifurcation on the plane. This enables us to perturb (5.3) as in the above equation so that it satisfies the desired eigenvalue condition while keeping the inclination-flip homoclinic orbits, since, when $\gamma = 0$, the existence of the inclination-flip homoclinic orbit in the entire three-dimensional space is equivalent to that of just a homoclinic orbit in the plane as explained in §6. Instead of using planar result, we have in this paper developed a Melnikov-like technique adapted to this situation in order to keep the inclination-flip homoclinic orbits under perturbation.

This task is necessary for our degenerate singularity result, because Rychlik's equation given above cannot be used for our purpose due to the lack of the essential term $x^2y\frac{\partial}{\partial y}$ after taking the scaling limit introduced in §5. For the same reason, two other equations given in [Robinson 1989, 1992] are inappropriate for our purpose as well. Furthermore, in the case of homoclinic orbits with resonant eigenvalues considered by Robinson, we do not have a good limit system which possesses this particular homoclinic orbit in the scaling limit, like the equation (5.3) in the inclination-flip case. Difficult in Robinson's case is to verify the transversality condition for the stable and extended unstable manifolds along the homoclinic orbit.

In any case, in order to be able to prove the existence of a geometric Lorenz attractor in these equations, including ours, it is needed to add one extra term to (2.1), and hence these results do not imply an analogous result for the original Lorenz equation (1.1), not even for small perturbations of it. However, we can show the following (somewhat strange) fact:

Proposition 10.1. *There exists an arbitrarily small perturbation of the Lorenz equation such that the perturbed system has a geometric Lorenz "repeller".*

Here the geometric Lorenz repeller is a time-reversed object of the geometric Lorenz attractor, and hence this Proposition says that the perturbed system has a geometric Lorenz attractor if the time axis is reversed.

Proof. As we have seen in §2, the original Lorenz equation can be put into the form (2.2) which seems to be a subsystem of (3.2). In order to put (2.2) in the same perturbation situation as (3.2), we need to assume that the coefficients εp and εq are both very small compared to $\varepsilon^2 a$ and $-\varepsilon b$, which implies

$$\sigma + 1 \approx 0 \qquad \text{and} \qquad 2\sigma - b \approx 0,$$

that is,

$$\sigma \approx -1 \qquad \text{and} \qquad b \approx -2.$$

Therefore we need to make $\varepsilon < 0$ in order for the parameter γ, corresponding to $-\varepsilon b$, being negative as required in Theorem 7.1, which means we need to reverse the time. Once we take $\varepsilon < 0$ and admit the time-reversal, then we easily see that the equation (3.2) can be regarded as a perturbation of (2.2) which is equivalent to the time-reversed Lorenz equation. Applying Theorem 7.1 to this situation, we obtain the conclusion. □

Unfortunately, this argument is not applicable to the original Lorenz equation but we need to perturb it. Therefore the question about the existence of a strange attractor (or even a strange repeller) for the Lorenz equation is still open, to the authors' knowledge.

References

[1] V.S.Afraimovich, V.V.Bykov and L.P.Shil'nikov, On structurally stable attracting limit sets of Lorenz attractor type, Trans. Moscow Math. Soc., 44 (1983), 153–216.

[2] V.S.Afraimovich and Ya.B.Pesin, Dimension of Lorenz type attractors, Sov. Sci. Rev., C:Math./Phys., 6 (1987), 169–241.

[3] S.-N.Chow, B.Deng and B.Fiedler, Homoclinic bifurcations at resonant eigenvalues, J. Dyn. Diff. Eq., 2 (1990), 177–244.

[4] J.Carr, *Applications of Centre Manifold Theory,* 1981, Springer.

[5] F.Dumortier, Local study of planar vector fields: singularities and their unfoldings, in *Structures in Dynamics,* H.Broer et al. (eds.), pp. 161–241, 1991, North-Holland.

[6] J.Guckenheimer, A strange strange attractor, in *Hopf Bifurcation and its Applications*, J.E.Marsden and M.McCracken (eds.), pp. 368–381, 1976, Springer.

[7] J.Guckenheimer and P.Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* 1983, 3rd Printing 1989, Springer.

[8] J.Guckenheimer and R.F.Williams, Structural stability of Lorenz attractors, Publ. Math. IHES, 50 (1979), 59–72.

[9] J.Gruendler, The existence of homoclinic orbits and the method of Melnikov for systems in $\mathbb{R}^n$, SIAM J. Math. Anal., 16 (1985), 907–931.

[10] M.Hirsch, C.Pugh and M.Shub, *Invariant manifolds,* Lect. Notes Math., Vol. 583, 1977, Springer.

[11] E.I.Horozov, Versal deformations of equivariant vector fields in the case of symmetry of order 2 and 3, Trudy Sem. I.G.Petrovskava, 5 (1979), 163–192 (in Russian).

[12] G.Keller Generalized bounded variation and applications to piecewise monotonic transformations, Z. Wahrsch. Verw. Gebiete 69 (1985), 461–478.

[13] M.Kisaka, H.Kokubu and H.Oka, Supplement to homoclinic doubling bifurcation in vector fields, in *Dynamical Systems, Santiago 1990,* R. Bamon et al. (eds.), pp. 92–116, Pitman Res. Notes, Vol. 285, 1993, Longman.

[14] M.Kisaka, H.Kokubu and H.Oka, Bifurcations to N-homoclinic and N-periodic orbits in vector fields, J. Dyn. Diff. Eq., 5 (1993), 305–357.

[15] H.Kokubu, Homoclinic and heteroclinic bifurcations of vector fields, Japan J. Appl. Math., 5 (1988), 455–501.

[16] H.Kokubu and H.Oka, Bifurcation of geometric Lorenz attractors from orbit-flip double homoclinic loops, in preparation.

[17] E.N.Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci., 20 (1963), 130–141.

[18] H.Oka, Generation of geometric Lorenz attractors from degenerate vector field singularities, to appear in Proceedings of the 7th TOYOTA Conference "Towards Harnessing of Chaos", 1994.

[19] J.Palis Jr. and W.de Melo, *Geometric Theory of Dynamical Systems: An Introduction,* 1981, Springer.

[20] D.Rand, The topological classification of Lorenz attractors, Math. Proc. Camb. Phil. Soc., 83 (1978), 451–460.

[21] C.Robinson, Differentiability of the stable foliation of the model Lorenz equations, in *Dynamical Systems and Turbulence*, D.A.Rand and L.-S.Young (eds.), pp. 302–315, Lect. Notes Math., Vol. 898, 1981, Springer

[22] C.Robinson, Transitivity and invariant measures for the geometric model of the Lorenz equations, Erg. Th. Dyn. Sys., 4 (1984), 605–611; Errata, ibid., 6 (1986), 323.

[23] C.Robinson, Homoclinic bifurcation to a transitive attractor of Lorenz type, Nonlinearity, 2 (1989), 495–518.

[24] C.Robinson, Homoclinic bifurcation to a transitive attractor of Lorenz type, II, SIAM J. Math. Anal., 23 (1992), 1255–1268.

[25] M.Rychlik, Lorenz attractors through Sil'nikov-type bifurcation: Part I, Erg. Th. Dyn. Sys., 10 (1990), 793–821.

[26] B.Sandstede, Verzweigungstheorie homokliner Verdopplungen, Thesis, University of Stuttgart, 1993.

[27] S.Ushiki, H.Oka and H.Kokubu, Existence d'attracteurs étranges dans le déploiement d'une singularité dégénérée d'un champ de vecteurs invariant par translation, C. R. Acad. Sc. Paris, t. 298 (1984), Série I, 39–42.

[28] A.Vanderbauwhede, Center manifolds, normal forms and elementary bifurcations, in *Dynamics Reported,* Vol.2, U.Kirchgraber and H.O.Walther (eds.), pp. 89–169, 1989, John Wiley.

[29] R.F.Williams, The structure of Lorenz attractors, Publ. Math. IHES, 50 (1979), 321–347.

[30] E.Yanagida, Branching of double pulse solutions from single pulse solutions in nerve axon equations, J. Diff. Eq., 66 (1987), 243–262.