New aspects in the unfolding of the nilpotent singularity of codimension three

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Abstract

This paper is concerned with three-dimensional vector fields and more specifically with the study of dynamics in unfoldings of the nilpotent singularity of codimension three. Our ultimate goal is to understand the dynamics and bifurcations in the unfolding of the singularity. However, it is clear from the literature that the bifurcation diagram is very complicated and a complete study is far beyond the current possibilities not only from a theoretical point of view but also from a numerical point of view, despite recent developments of computational methods for dynamical systems. Since all complicated dynamical behavior is known to be of small amplitude, shrinking to the singularity for parameter values tending to the bifurcation parameter, our aim in this paper is especially to focus on a different aspect that might be interesting in the study of global bifurcation problems in the presence of such a nilpotent singularity of codimension three. We introduce the notion of *traffic regulator* and the specific sets called the *inset* and *outset*, which give a new framework for studying a transition map in a cylindrical neighborhood of the singularity that contains all the nontrivial dynamics that can bifurcate from the singularity, focusing on the domains on which the transition map is defined.

We also give a list of open problems which we believe to be helpful for future investigation of the bifurcations from the nilpotent triple zero singularity in $\mathbb{R}^3$.

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1 Introduction

This paper deals with three-dimensional vector fields and more specifically is concerned with the study of dynamics in unfoldings of the nilpotent singularity of codimension three. The ultimate goal of the study is to understand the dynamics and bifurcations that occur in the unfoldings of the singularity. A generic nilpotent singularity of codimension two in the plane is known as the Bogdanov-Takens singularity and its structure and unfoldings are fully understood ([1], [37], [9] and references therein). Singularities with more degenerate nonlinear terms have been studied well and their bifurcation structures have thoroughly been investigated ([13], [14]). The singularities that are studied in this paper are natural three-dimensional analogues of those, and the topological types of the singularities have already been classified completely up to codimension four ([10]). In this paper we attempt to study the bifurcations from such singularities having the least degenerate nonlinear terms.

For sure it would be interesting to know the complete bifurcation diagram and related dynamical patterns. It is, however, clear from the literature (see, e.g. [19], [31]) that the bifurcation diagram is very complicated and a complete study is far beyond the current possibilities not only from a theoretical point of view but also from a numerical point of view, despite recent development of computational methods for dynamical systems such as HomCont/AUTO and GAIO among others; see also the discussion in §6.5. From [10], it is at least clear that all complicated dynamical behavior is of small amplitude, shrinking to the singularity for parameter values tending to the bifurcation parameter. Besides providing some extra information about the bifurcation patterns, our aim in this paper is especially to focus on a different aspect that might be interesting in the study of global bifurcation problems in the presence of such a nilpotent singularity of codimension three. Indeed the nilpotent singularity could, for instance, belong to a (non-hyperbolic) homoclinic loop, near which interesting global dynamics could be created. This is very comparable to the cuspidal loop in the plane as studied in [15]. The way to study such a loop is based on a detailed study of the transition map near the nilpotent singularity using only partial information of the Bogdanov-Takens bifurcation. In this paper we start the study of a similar transition map in the three-dimensional case, focusing on the domains on which the transition map is defined.

We introduce the notion of *traffic regulator* and the specific sets associated to it called the *inset* and *outset*. This provides us with a new framework for studying a transition map in a cylindrical neighborhood of the singularity containing all the non-trivial dynamics that can bifurcate from the singularity. The inset and outset are the complements of the domains of definition of the transition map on each of the transitive disks of the cylindrical neighborhood. One may consider these sets as shadows of the maximal invariant set sitting inside the neighborhood. They hence carry some information about the invariant set itself, although being subsets of the 2-disk and hence easier to be handled. We study topological properties of these sets and will describe the structure of them in part of the parameter region of the unfolding. The structure of the sets becomes more complicated in other parameter regions, and we shall also discuss some of these cases based on numerical experiments.

The organization of the paper is as follows. In §2, we give the precise definition of the singularity and the unfoldings which will be studied, together with a brief summary of basic results used along the paper, most of them known in the literature. §3 contains the study of dynamics and bifurcations in a part of the parameter space where the corresponding system has a Lyapunov function. In §4, we introduce the new idea of looking at the dynamics by defining the notion of traffic regulator as well as the inset and outset. It will provide partial information about the maximal compact invariant set of the system. We shall describe how these inset and outset change under variation of the unfolding parameters. In §5, we shall focus on the specific codimension one subset in the parameter space where the corresponding system is divergence-free. We believe that the study of the dynamics in this particular region will be one of the crucial steps toward a complete understanding of the entire bifurcation diagram. Finally in §6, we list a number of unsolved problems for future investigation. We consider this section as an important part of this paper, because we believe that these open problems provide a good summary of the current stage of the study of this nilpotent singularity.

2 Preliminaries

In this section we first recall some of the results obtained in [10] about nilpotent singularities in $\mathbb{R}^3$ and state a new property, essential in our paper, which can be proved by means of the techniques used there. We will also summarize some other basic facts about the unfolding.

2.1 Singularity

Consider a C^∞ vector field X defined in a neighborhood of $O \in \mathbb{R}^3$. We assume that X vanishes at O and the 1-jet of X at O is linearly conjugate to

$$y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

As is proven in [10], up to a C^∞ coordinate change, X can be given by the following normal form:

$$y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (ax^2 + bxy + cxz + dy^2 + O(\|(x,y,z)\|^3))\frac{\partial}{\partial z}. \tag{2.1}$$

The condition $a \neq 0$ defines a stratum of codimension three in the space of germs of C^∞ vector fields with a singularity at the origin, in which only one topological type is given. More precisely,

Theorem 2.1 ([10]) *Let X be a singularity as in (2.1) with $a \neq 0$. Then at the origin X is locally C^0-equivalent to*

$$y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}.$$

Throughout this paper, we call a singularity as in (2.1), with $a \neq 0$, the *nilpotent singularity of codimension three.*

2.2 Unfolding

According again to [10], any generic three parameter family X_γ, with $\gamma = (\lambda, \mu, \nu) \in \mathbb{R}^3$, that unfolds the nilpotent singularity of codimension three admits the normal form

$$\begin{aligned} y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} &+ (\lambda + \mu y + \nu z + x^2 + \\ &bxy + cxz + dy^2 + eyz + \alpha(x,y,z,\gamma))\frac{\partial}{\partial z}, \end{aligned} \tag{2.2}$$

where

$$\alpha(x, y, z, \gamma) = O(\|(x, y, z, \lambda, \mu, \nu)\|^3) = O(\|(y, z)\|)$$

and λ, μ, ν represent the exact coefficients in the Taylor expansion with respect to (x, y, z), namely the following holds:

$$\alpha(0, \gamma) = \frac{\partial \alpha}{\partial y}(0, \gamma) = \frac{\partial \alpha}{\partial z}(0, \gamma) = 0.$$

The unfolding X_γ will be studied by using the rescaling

$$\lambda = u^6 \bar{\lambda}, \quad \mu = u^2 \bar{\mu}, \quad \nu = u \bar{\nu},$$
$$x = u^3 \bar{x}, \quad y = u^4 \bar{y}, \quad z = u^5 \bar{z},$$

where $\bar{\lambda}^2 + \bar{\mu}^2 + \bar{\nu}^2 = 1$ and $(\bar{x}, \bar{y}, \bar{z}) \in A$, a fixed ball in $\mathbb{R}^3$. Thus we obtain

$$\bar{y}\frac{\partial}{\partial \bar{x}} + \bar{z}\frac{\partial}{\partial \bar{y}} + (\bar{\lambda} + \bar{\mu}\bar{y} + \bar{\nu}\bar{z} + \bar{x}^2 + O(u))\frac{\partial}{\partial \bar{z}}. \tag{2.3}$$

The next theorem states that all the interesting phenomena that occur in a neighborhood of the singularity can be obtained by means of the previous rescaling.

Theorem 2.2 ([10]) *Let $C = D^2 \times [-1, 1]$ where D^2 stands for the standard 2-disk in $\mathbb{R}^2$. There exist a neighborhood $W \subset \mathbb{R}^3$ of $\gamma = O$ and a continuous map*

$$\Psi : C \times W \to \mathbb{R}^3$$

such that, for all $\gamma \in W$, $V_\gamma = \Psi(C, \gamma)$ is a neighborhood of $0 \in \mathbb{R}^3$ homeomorphic to C and

(1) *$\Psi(S^1 \times [-1, 1], \gamma)$ consists of regular orbits of X_γ,*

(2) *$\Psi(D^2 \times \{-1, 1\}, \gamma)$ is transverse to X_γ with X_γ pointing inward along $\Psi(D^2 \times \{-1\}, \gamma)$ and outward along $\Psi(D^2 \times \{1\}, \gamma)$.*

Moreover the restricted vector field $X_\gamma|_{V_\gamma}$ has the following properties:

(i) *In case $\lambda > 0$, $X_\gamma|_{V_\gamma}$ is a flow box.*

(ii) *In case $\lambda = 0$, the nonwandering set $\Omega(X_\gamma|_{V_\gamma})$ consists of a unique equilibrium point.*

(iii) *In case* $\lambda < 0$, $X_\gamma|_{V_\gamma}$ *has two equilibrium points and* $\Omega(X_\gamma|_{V_\gamma})$ *is contained in*

$$A_v = \{(v^3\bar{x}, v^4\bar{y}, v^5\bar{z}) \mid (\bar{x}, \bar{y}, \bar{z}) \in A\}$$

with some v *satisfying* $\gamma = (v^6\bar{\lambda}, v^2\bar{\mu}, v\bar{\nu})$ *and* A *being a fixed open ball in the* $(\bar{x}, \bar{y}, \bar{z})$ *space.*

Although it follows from the same arguments used in the proof of the previous result, the next theorem was not explicitly stated in [10]. For this reason and since it is essential in our paper, we include it, together with a short proof.

Theorem 2.3 *Consider the* S^2 *family* $Y_{\bar{\gamma}}$ *given by (2.3) with* $u = 0$. *There exists a continuous map*

$$\bar{\Psi} : C \times S^2 \to A \subset \mathbb{R}^3,$$

where $C = D^2 \times [-1, 1]$, *such that, for all* $\bar{\gamma} \in S^2$, *the domain* $V_{\bar{\gamma}} = \bar{\Psi}(C, \bar{\gamma})$ *is homeomorphic to* C *and*

(1) $\bar{\Psi}(S^1 \times [-1, 1], \bar{\gamma})$ *consists of regular orbits of* $Y_{\bar{\gamma}}$;

(2) $\bar{\Psi}(D^2 \times \{-1, 1\}, \bar{\gamma})$ *is transverse to* $Y_{\bar{\gamma}}$ *with* $Y_{\bar{\gamma}}$ *pointing inward along* $\bar{\Psi}(D^2 \times \{-1\}, \bar{\gamma})$ *and outward along* $\bar{\Psi}(D^2 \times \{1\}, \bar{\gamma})$;

(3) *The nonwandering set* $\Omega(Y_{\bar{\gamma}})$ *and even the maximal compact invariant set of* $Y_{\bar{\gamma}}$ *are contained in the interior of* $V_{\bar{\gamma}}$.

Moreover $Y_{\bar{\gamma}}|_{V_{\bar{\gamma}}}$ *has the following properties:*

(i) *If* $\bar{\lambda} > 0$, $Y_{\bar{\gamma}}|_{V_{\bar{\gamma}}}$ *is a flow box;*

(ii) *If* $\bar{\lambda} = 0$, $\Omega(Y_{\bar{\gamma}}|_{V_{\bar{\gamma}}})$ *consists of a unique equilibrium point.*

Proof. To prove the result we use the technique of family blowing-up (details about this technique can be found in [10]). We first consider curves in the parameter space defined by

$$(\lambda, \mu, \nu) = (v^6\Lambda, v^2M, vN)$$

with $\Gamma = (\Lambda, M, N) \in S^2$ and $v \in [0, \infty)$. Second we apply to (2.2) a quasi-homogeneous blowing-up given by

$$(v, x, y, z) = (u\bar{v}, u^3\bar{x}, u^4\bar{y}, u^5\bar{z}),$$

with $\bar{v}^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = 1$ and $u \in [0, \infty)$. Thus, we obtain a C^∞ family of vector fields $\bar{Y}_\Gamma$ on $S^3 \times [0, \infty)$. In the sequel we denote $\bar{p} = (\bar{x}, \bar{y}, \bar{z})$.

The vector field $\bar{Y}_\Gamma$ is studied by means of different charts on S^3. Taking $\bar{p} \in S^2$ and $\bar{v} \in \Delta$, where Δ is a small neighborhood of $0 \in \mathbb{R}$, we obtain a C^∞ family of vector fields $\bar{Y}^1_\Gamma$, defined on $S^2 \times \Delta \times [0, \infty)$. On the other hand, taking $\bar{p} \in A$ and $\bar{v} = 1$, where $A \subset \mathbb{R}^3$ is an arbitrarily large compact set, we obtain a C^∞ family of vector fields $\bar{Y}^2_\Gamma$, defined on $A \times [0, \infty)$. Note that $\bar{Y}^2_\Gamma$ is nothing but the family (2.3) in which we take $\bar{\gamma} = (\bar{\lambda}, \bar{\mu}, \bar{\nu}) = (\Lambda, M, N) = \Gamma$.

The vector field $\bar{Y}^1_\Gamma$ restricted to $S^2 \times \Delta \times \{0\}$ gives the behavior at infinity of $\bar{Y}^2_\Gamma$ restricted to $A \times \{0\}$. On the other hand, taking different charts on S^2, it turns out that $\bar{Y}^1_\Gamma$ presents only two hyperbolic equilibrium points on $S^2 \times \Delta \times \{0\}$ at $E_1 = (\bar{p}_1, 0, 0)$ and $E_2 = (\bar{p}_2, 0, 0)$. Moreover, on $S^2 \times \{0\} \times \{0\}$ the vector field is Morse-Smale with no periodic orbits and E_1 and E_2 are, respectively, a sink and a source. On the other hand, $W^s(E_1)$ and $W^u(E_2)$ are three-dimensional invariant manifolds contained in $S^2 \times \Delta \times \{0\}$.

Let us consider fundamental domains D^s and D^u of $W^s(E_1)$ and $W^u(E_2)$, respectively. Since the flow of $\bar{Y}^1_\Gamma$ restricted to $S^2 \times \{0\} \times \{0\}$ connects D^u and D^s, we can take, on $S^2 \times \Delta \times \{0\}$, open neighborhoods V^s and V^u of $D^s \cap (S^2 \times \{0\} \times \{0\})$ and $D^u \cap (S^2 \times \{0\} \times \{0\})$, respectively, such that $\partial(D^s \setminus V^s)$ and $\partial(D^u \setminus V^u)$ are also connected by the flow of $\bar{Y}^1_\Gamma$ along a hypersurface $T \subset S^2 \times \Delta \times \{0\}$.

Let us take

$$\bar{V} = (D^s \setminus V^s) \cup (D^u \setminus V^u) \cup T.$$

We can assume that A is large enough to get $\bar{V}$ covered by the chart $A \times \{0\}$. This leads to the existence of the S^2 family of sets $V_{\bar{\gamma}}$ satisfying (1), (2) and (3).

Properties (i) and (ii) follow from the existence of a Lyapunov function, a function which increases along orbits, given by $L(\bar{x}, \bar{y}, \bar{z}) = \bar{z} - \bar{\mu}\bar{x} - \bar{\nu}\bar{y}$ and from a simple local analysis of the singularity at the origin when $\bar{\lambda} = 0$. $\square$

Remark 2.4 The existence of $V_{\bar{\gamma}}$ will be used in §4 in order to introduce the notion of inset and outset.

2.3 Preliminary study of the limit system

Since all the dynamics in the unfolding are detectable by the family rescaling (2.3), our first objective should be the study of the S^2 family in (2.3) for $u = 0$, namely,

$$\begin{aligned} \bar{x}' &= \bar{y} \\ \bar{y}' &= \bar{z} \\ \bar{z}' &= \bar{\lambda} + \bar{\mu}\bar{y} + \bar{\nu}\bar{z} + \bar{x}^2, \end{aligned} \tag{2.4}$$

where $\bar{\lambda}^2 + \bar{\mu}^2 + \bar{\nu}^2 = 1$. In this section we pay attention to the local bifurcations which are present in such family.

If $\bar{\lambda} > 0$, we have no equilibrium points and the vector field behaves as a flow box. On the other hand, (2.4) is invariant under the transformation

$$(\bar{\lambda}, \bar{\mu}, \bar{\nu}, \bar{x}, \bar{y}, \bar{z}, t) \to (\bar{\lambda}, \bar{\mu}, -\bar{\nu}, -\bar{x}, \bar{y}, -\bar{z}, -t),$$

and therefore we only consider (2.4) for values of the parameter in the region

$$R = \{(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in S^2 \mid \bar{\lambda} \leq 0,\ \bar{\nu} \leq 0\}.$$

For parameter values in the region R, (2.4) has a unique equilibrium point at $(0,0,0)$ when $\bar{\lambda} = 0$ and two equilibrium points at $P_- = (-\sqrt{-\bar{\lambda}}, 0, 0)$ and $P_+ = (\sqrt{-\bar{\lambda}}, 0, 0)$ when $\bar{\lambda} < 0$. With the study of the linear part at the equilibrium points we detect immediately some cases of degeneracy. All the subsequent local bifurcations are summarized in the following theorem.

Theorem 2.5 *In the region R we distinguish the following local bifurcations (see Figure 1).*

(1) *Along the curve $SN1 = \{(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in R \mid \bar{\lambda} = 0, \bar{\mu} < 0\}$ the linear part at $(0,0,0)$ has two eigenvalues with negative real part and one eigenvalue equal to 0. Moreover two hyperbolic equilibrium points P_- and P_+ are created by a generic saddle node bifurcation, where P_- is an attractor and P_+ is a saddle with the stability index 2, namely, $\dim W^s(P_+) = 2$.*

(2) *Along the curve $SN2 = \{(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in R \mid \bar{\lambda} = 0, \bar{\mu} \geq 0\}$ the linear part at $(0,0,0)$ has two real eigenvalues with opposite sign and one eigenvalue equal to 0. Moreover two hyperbolic equilibrium points P_- and P_+ are created by a generic saddle node bifurcation, which are saddles with the stability index 1 and 2, respectively.*

(3) *There exists a curve* $H \subset R$ *joining* $(0,-1,0)$ *and* $(0,0,-1)$*, given by the equation* $\bar{\lambda} = -\bar{\mu}^2\bar{\nu}^2/4$ *with* $\bar{\lambda} < 0$ *and* $\bar{\nu} < 0$*, along which the linear part at* P_- *has eigenvalues* $\{\alpha, i\omega, -i\omega\}$*, with* $\alpha < 0$ *and* $\omega > 0$*. On* H *the family (2.4) undergoes a generic Hopf bifurcation, which creates a hyperbolic attracting limit cycle.*

(4) *At* $\bar{\gamma} = BT = (0,0,-1)$*, where the linear part at* $(0,0,0)$ *has eigenvalues* $\{0,0,-1\}$*, the family (2.4) undergoes a generic Bogdanov-Takens bifurcation. The limit cycle that bifurcates along* H *disappears in a homoclinic bifurcation along the curve Hom, see Figure 1.*

Figure 1: Local bifurcations and curves D^+ and D^-, where two eigenvalues at P_+ and P_-, respectively, change from real to complex. The parameter HZ, for which the system has a Hopf-Zero singularity, and the divergence zero curve DZ are also depicted. Note that the region R is shown by means of its projection onto the (μ, ν)-plane.

Remark 2.6 For all parameter values in R, with $\bar{\lambda} < 0$, the equilibrium point P_+ is hyperbolic and the correspondent linear part has two eigenvalues with negative real part and one real positive eigenvalue. On the other hand H divides R into two open regions R_1 and R_2 (see Figure 1). For parameters in those regions P_- is hyperbolic and the correspondent linear part has all the eigenvalues with negative real part on R_1 and two eigenvalues with positive real part and one real negative eigenvalue on R_2. In Figure 1 we also indicate the position of the curves in R where the eigenvalues change from real to complex.

Remark 2.7 The techniques used to prove the existence of the local bifurcations described in Theorem 2.5 are standard: reduction to center manifolds and also reduction to normal form in order to show the genericity of the correspondent unfoldings. For this reason we prefer not to include a proof. Moreover, similar results can be found in the literature (see, for instance, [28] and [17]).

At $\bar{\gamma} = HZ = (0, -1, 0)$ (see Figure 1) the linear part at $(0, 0, 0)$ has eigenvalues $\{0, i, -i\}$. A simple normal form analysis permits to check that, according with [36], such singularity is of codimension two and topologically equivalent to the vertical vector field $(x^2+y^2+z^2)\frac{\partial}{\partial z}$. This type of degeneracy is called the Hopf-Zero singularity.

A complete understanding of all the different bifurcations which can appear in the unfolding of the Hopf-Zero singularity has not yet been achieved. In [18], [20] and [39], the symmetric unfoldings were studied. The occurrence of Shil'nikov type homoclinic orbits was discussed in [20] and [3]. In this last paper their existence was proven for generic families and the C^∞ flatness character of the phenomenon was pointed out.

For generic two-parameter families that unfold the Hopf-Zero singularity, it is known that the stable periodic orbit which is generated after the Hopf bifurcation becomes unstable and a stable torus is born from it. This phenomenon occurs along a bifurcation curve and the invariant tori persist for values of the parameters in an open region. Note that, since the divergence of family (2.4) is equal to a constant $\bar{\nu}$, it cannot exhibit invariant tori unless $\bar{\nu} = 0$ which is not an open condition for the parameters, and hence our family (2.4) is not a generic unfolding of the Hopf-Zero singularity in that sense. Nevertheless, one can check that the influence of the terms of order $O(u)$ in (2.3) permits to show that the original family (2.2) may exhibit invariant tori in an open region of the bifurcation diagram.

3 Dynamics when $\mu \geq 0$

We first observe that the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= \lambda + \mu y + \nu z + x^2 \end{aligned} \tag{3.1}$$

has a Lyapunov function

$$L(x, y, z) = yz - \frac{1}{3}x^3 - \lambda x - \frac{1}{2}\nu y^2, \tag{3.2}$$

provided that $\mu \geq 0$. In fact, a simple calculation shows

$$\frac{d}{dt}L(x(t), y(t), z(t)) = z(t)^2 + \mu y(t)^2$$

for a solution $(x(t), y(t), z(t))$ of (3.1). Therefore the value of L along an orbit is non-decreasing everywhere and is strictly increasing except along the x-axis (the (x, y)-plane if $\mu = 0$).

Theorem 3.1 *If $\lambda < 0$ and $\mu \geq 0$, there exists at least one heteroclinic orbit from the equilibrium point $P_- = (-\sqrt{-\lambda}, 0, 0)$ to $P_+ = (\sqrt{-\lambda}, 0, 0)$.*

PROOF. Notice that the heteroclinic orbit, if it exists, lies in a non-trivial intersection of two-dimensional invariant manifolds $W^u(P_-)$ and $W^s(P_+)$. Suppose that $W^u(P_-) \cap W^s(P_+) = \emptyset$. Let $l \subset W^s(P_+)$ be a closed curve contained in the set V_γ such that

$$W^s(P_+) = \{\varphi(t, p) \mid t \in \mathbb{R},\ p \in l\} \cup \{P_+\},$$

namely l is a fundamental domain of $W^s(P_+)$. Given $p \in l$, its negative orbit cuts $D_{in} = \Psi(D^2 \times \{-1\}, \gamma)$. In fact, otherwise, the negative orbit of p would be bounded and hence $\alpha(p) \neq \emptyset$. Note that $\alpha(p)$ is either P_+ or P_-, since these are the only invariant sets contained in $\{p \in V_\gamma \mid \dot{L} = 0\}$. Since we are assuming that $W^u(P_-) \cap W^s(P_+) = \emptyset$, $\alpha(p) \neq P_-$. However, $\alpha(p) \neq P_+$, because a homoclinic orbit of P_+ cannot exist if we have a Lyapunov function. Therefore $W^s(P_+)$ must cut the set D_{in} along a closed curve γ. All the orbits starting in the interior of γ stay in V_γ for all $t \geq 0$.

This implies that $\omega(p) = P_+$ for all $p \in C_\gamma$ where C_γ is the region enclosed by the closed curve γ. This contradicts the fact that P_+ and P_- are hyperbolic equilibrium points of saddle-type. The existence of at least one heteroclinic orbit from P_- to P_+ is thus proved. $\square$

We shall study the invariant sets in more detail by using the level surfaces of the Lyapunov function L. Notice that the gradient of the function L is given by

$$\mathrm{grad}L(x, y, z) = (-x^2 - \lambda, z - \nu y, y),$$

and hence the critical points of L are only at $P_\pm$. Let $c_\pm = L(P_\pm) = \pm\frac{2}{3}(-\lambda)^{3/2}$. The Hessian at each of the critical points $P_\pm$ is given by

$$\mathrm{Hess}L(P_\pm) = \begin{pmatrix} \mp 2\sqrt{-\lambda} & 0 & 0 \\ 0 & -\nu & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, if $\lambda < 0$, it has two positive and one negative eigenvalues at P_-, while two negative and one positive eigenvalues at P_+, and hence L is a Morse function. Since L is increasing along orbits, negative eigenvalues correspond to the stable manifolds of $P_\pm$, while positive eigenvalues to the unstable manifolds.

Let $V = V_\gamma$ be the box constructed in Theorem 2.3, and consider the level surface given by $S_c = L^{-1}(c) \cap V$. One can choose it in such a way that the top and bottom boundaries $D_{in} = \Psi(D^2 \times \{-1\}, \gamma)$ and $D_{out} = \Psi(D^2 \times \{1\}, \gamma)$ lie entirely in level surfaces $S_{c_{in}}$ and $S_{c_{out}}$, respectively. Studying the blowing-up of the family, it follows that the x-coordinates of the repeller [resp. the attractor] at infinity are negative [resp. positive]. Therefore one can take D_{in} and D_{out} as 2-disks whose x-coordinate is negative for D_{in} and positive for D_{out}.

Let us consider how the topology of the level surface S_c changes as c varies from $c_{in} < 0$ to $c_{out} > 0$ through the critical values $c_\pm$. From above, $S_{c_{in}}$ is a 2-disk whose x-coordinates are negative with sufficiently large absolute value. Since the flow is monotone on the side boundary $K = \Psi(S^1 \times [-1, 1], \gamma)$ of the box V which is homeomorphic to a cylinder, the intersection of K and S_c for any c is a circle. Let K_c be the subset of K which consists of the points p in K at which $L(p) \geq c$. Clearly K_c is again a cylinder except for $c = c_{out}$ in which case it is collapsed to a circle. It will be convenient to consider a closed surface M_c given by

$$M_c = S_c \cup K_c \cup D_{out}.$$

Obviously $M_{c_{in}}$ is nothing but the boundary of V which is a 2-sphere.

Due to the Morse theory, the topological type of S_c, and hence of M_c, changes only when L takes a critical value. Therefore we have only to study it around its critical values $c_\pm$. M_c is diffeomorphic to a 2-sphere for $c < c_- = -\frac{2}{3}(-\lambda)^{3/2}$. At $c = c_-$, the equilibrium point P_- is a unique critical point in this level and has the Morse index one. Note that here the Morse index is defined as the number of negative eigenvalues of the Hessian at the critical point. According to the Morse lemma, local structure of the level surfaces near the critical point can be described by those of the quadratic form $x^2 + y^2 - z^2$, and hence it changes from a two-sheeted hyperboloid to a one sheeted hyperboloid, as c passes the critical value c_- from below. This change of local structure of the level surfaces occurs along the stable manifold of the critical point (equilibrium point) P_-, and therefore the surface S_{c_-} at the critical level becomes a pinched 2-disk at the critical point and then it changes to a surface with a neighborhood of the critical point replaced by a one-sheeted hyperboloid for c slightly larger than c_-. See Figure 2. This makes the closed surface M_c change from a 2-sphere to a 2-torus which encloses the critical point P_- inside. Moreover, the intersection of the local unstable manifold of P_- with M_c for c sufficiently close to, but larger than, c_- is a simple closed curve which can be considered as the meridian of the torus M_c.

As c varies from c_- to c_+, the surface M_c remains a 2-torus. However, the intersection of the unstable manifold of P_- could become complicated, although its isotopy class does not change. As c increases to the other critical value c_+, the surface S_c, and hence M_c, comes closer to P_+, and at $c = c_+$, the surface S_c is again pinched at P_+. Here the Morse index is two, and hence local structure of the level surface near P_+ changes from a one-sheeted hyperboloid to a two-sheeted hyperboloid as c crosses c_+. Thus S_c changes to a 2-disk, and hence M_c turns from a 2-torus back to a 2-sphere. See Figure 2. Notice, again, that when $c_- < c < c_+$ is sufficiently close to c_+, the intersection of the local stable manifold of P_+ with the surface S_c is a simple circle which can be considered as its longitude. After crossing this level, the topological type of the surface S_c does not change, and hence M_c remains to

Figure 2: Change of level surfaces.

be a 2-sphere for $c_+ < c < c_{out}$, and at $c = c_{out}$, it collapses to a 2-disk, as $S_{c_{out}}$ is chosen in the level surface of L.

Since the flow is always transverse to the level surface S_c for any c with $c_- < c < c_+$, the isotopy type of the intersection of $W^u(P_-)$ and $W^s(P_+)$ on the 2-torus M_c does not change. Since $W^u(P_-)$ is isotopic to the meridian and $W^s(P_+)$ is isotopic to the longitude on the 2-torus, it implies that the intersection of $W^u(P_-)$ and $W^s(P_+)$ on the 2-torus is unavoidable. This is an alternative proof of Theorem 3.1 for the existence of at least one heteroclinic connection from P_- to P_+. Moreover, the intersection number of $W^u(P_-)$ and $W^s(P_+)$ on the 2-torus can be made equal to one by an appropriate choice of orientations of the curves on the 2-torus.

It is likely that the intersection of these manifolds is unique for any value of the parameters, but the above argument cannot exclude the possibility of having more complicated intersection of these manifolds. However, since the system we consider is gradient-like, the invariant set in V is completely given by the orbits of intersection of $W^u(P_-)$ and $W^s(P_+)$ as well as the equilibria P_- and P_+.

4 Traffic regulator and its inset and outset

In Theorem 2.3, we have shown that there exists a box V which has the following properties:

(1) The boundary of V consists of D_{in}, D_{out}, and K, where the vector field is transverse inward on D_{in}, outward on D_{out}, whereas K is homeomorphic to a cylinder $S^1 \times [-1, 1]$ and it is a flow box;

(2) The maximal invariant set of the vector field in V is contained in its interior.

The box V is an isolating neighborhood of the maximal compact invariant set of the vector field (3.1) and has a convenient structure for describing the dynamics in a neighborhood of the invariant set, which we formulate as follows with the name 'traffic regulator'.

Definition 4.1 A *traffic regulator* in $\mathbb{R}^n$ is a domain V whose boundary consists of (1) an *entrance disk* D_{in}, (2) an *exit disk* D_{out}, and (3) a *boundary flow box* K. The entrance disk is an $(n-1)$-disk on which the vector field is transverse to the disk and points inward, whereas it is transverse and points outward on the exit $(n-1)$-disk. The vector field gives a flow box in the rest of the boundary K which is homeomorphic to $S^{n-2} \times [0, 1]$ and is called the boundary flow box. A positive orbit starting at a point in the entrance disk either stays in the domain forever or leaves the domain from the exit disk in finite time. In the former case, the initial point of the positive orbit belongs to the *inset* C_{in}, and in the latter case, it belongs to an open set in D_{in} called the *transit domain* and the time which the orbit spends in the domain V is called the *transit time*. For the exit disk, one can similarly define the *outset* C_{out} and the *transit domain* as well as the transit time. There is a natural one-to-one correspondence map called the *transition map* between the transit domains in the entrance disk and the exit disk, which is clearly a diffeomorphism, as smooth as the vector field, and the transit time is the same for a point in the entrance transit domain and the corresponding point in the exit transit domain.

A usual flow box in $\mathbb{R}^n$ is a trivial traffic regulator with empty inset and outset and the transition map is an everywhere defined diffeomorphism. Clearly, a traffic regulator can be defined to have more general entrance and exit domains rather than just one disk. For instance, a traffic regulator for a planar hyperbolic saddle point has the entrance domain consisting of two disjoint 1-disks and the same for the exit domain. However, Definition 4.1 is sufficient for our purpose in this paper, since V can be chosen as in Theorem 2.3.

In the study of planar vector fields, the transition map near a singularity carries very important information for the understanding of global dynamics, since in that case, the inset and outset are mostly one point, and hence the main information of the traffic regulator is contained in the transition map. For vector fields in dimension strictly larger than two, the nature of the transition near a singularity could be much more complicated, and the inset and outset are no longer simple. Therefore one must study the structure of inset and outset first to have a better understanding of the transition map. The transition time also carries information, as it goes to infinity when the point in $D_{in} \setminus C_{in}$ approaches C_{in}, and hence the set of points in D_{in} whose transit time is larger than a given constant defines a neighborhood of C_{in}. This set gives an outer approximation of the inset which can be obtained by means of a computational method as we do at the end of this section where we show some examples. It should also be noted that the method of Shil'nikov ([5] and references therein), which is useful to describe the transition of orbits that pass near a hyperbolic equilibrium point, has some similarity to the traffic regulator in the sense that the description of orbits is given in terms of in-coming and out-going sections as well as the transit time information between them.

In this section, we mainly study the inset and outset of the system (3.1). We also give general properties of these sets from topological point of view as well as their numerical examples. Clearly C_{in} and C_{out} are compact subsets of the interior of D_{in} and D_{out}, respectively. They are non-empty if and only if there is a non-empty invariant set in V, namely in the case $\lambda \leq 0$. Hereafter we always assume $\lambda \leq 0$.

4.1 The inset and outset for (3.1)

The goal of this subsection is to describe the structure of the inset and outset in a part of the parameter space for the system (3.1). First we give a complete description of the inset and outset near the saddle-node bifurcations of singularities.

Theorem 4.2 *Consider the unit sphere S^2 of the parameter space for $\gamma = (\lambda, \mu, \nu)$, $\lambda \leq 0$ and $\nu \leq 0$. Let $HZ = (0, -1, 0)$ and $BT = (0, 0, -1)$ be the Hopf-Zero and the Bogdanov-Takens bifurcation points on the unit sphere. Define $SN = \{\gamma \in S^2 \mid \lambda = 0\}$, and let Hom be the homoclinic bifurcation curve emanating from BT on the unit sphere. Then there exists a one-sided closed neighborhood R of the curve $SN \setminus HZ$ on the sphere which is divided into two regions by the curve Hom, such that the inset and outset for the parameters from the set R can be given as follows:*

(1) *At HZ, the inset and outset are both one point;*

(2) *At BT, the inset is a closed segment and the outset is a point;*

(3) *On Hom, the inset is the union of two 2-disks connected by a segment, while the outset is a point;*

(4) *In the region R_1 of R divided by Hom containing HZ in its boundary, the inset is a 2-disk, while the outset is a point;*

(5) *In the other region R_2 of R, both the inset and outset are closed segments.*

Moreover, in the region $R \setminus$ Hom, the inset and outset change continuously with respect to the Hausdorff metric. At Hom $\setminus BT$, the outset is continuous but the inset is not lower semicontinuous. See Figure 3.

Figure 3: Inset and outset close to $\lambda = 0$.

The proof of this theorem is rather lengthy and hence will be given in Appendix.

Remark 4.3 As proven in Theorem 4.2, the inset $C_{in}(\gamma)$ as a set-valued function changes discontinuously at the curve *Hom*. However, it is upper-semicontinuous, see Proposition 4.6.

In the case of $\mu \geq 0$ where we have the Lyapunov function, the inset and outset are relatively simple, although we cannot describe them completely, nor can we describe the structure of the invariant set $\mathrm{Inv}(V)$. If one knows that the heteroclinic orbit from P_- to P_+ is unique, which is the case when $\mu \geq 0, \nu = 0$ ([27, 38]), the inset and outset are both a segment, homeomorphic to a closed interval. The boundary points of the segment C_{in} correspond to $W^s(P_-) \cap D_{in}$ and the remaining part corresponds to $W^s(P_+) \cap D_{in}$. The same holds for C_{out} if $W^s(P_-)$ and $W^s(P_+)$ are replaced by $W^u(P_+)$ and $W^u(P_-)$, respectively. For $\nu < 0$ and $\mu \geq 0$, the inset and outset may become more complicated, but one can show that they at least share a common property called the accessibility as described in the following definition.

Definition 4.4 A point in the inset C_{in} is called *accessible* from ∂D_{in}, if there exists a continuous path connecting the point and ∂D_{in} in $D_{in} - C_{in}$. Accessibility of a point in the outset C_{out} is similarly defined.

Theorem 4.5 *When $\mu \geq 0$ and $\lambda < 0$, the one-dimensional stable manifold $W^s(P_-)$ of the equilibrium point P_- intersects D_{in} at two points. These points belong to C_{in} and are accessible from the boundary ∂D_{in}. Similarly $W^u(P_+)$ intersects D_{out} at two points in C_{out} and these points are accessible from ∂D_{out}.*

PROOF. As discussed in §3, the invariant set in V consists of the equilibria $P_\pm$ and their connecting orbits, provided that $\mu \geq 0$, namely the system is gradient-like. In this case the inset consists of the intersection of the stable manifolds of $P_\pm$ with D_{in}. In particular there exist two points of intersection of the one-dimensional stable manifold $W^s(P_-)$. Therefore, if one of these two points is not accessible, it must be enclosed by part of the two-dimensional stable manifold $W^s(P_+)$ in D_{in}.

Recall from §3 that if the Lyapunov level c satisfies $c_- < c < c_+$, then the corresponding surface M_c is a 2-torus, on which $W^s(P_+)$ and $W^u(P_-)$ intersect in such a way that the intersection number is equal to one. If we take $c > c_-$ close enough to c_-, the intersection of $W^u(P_-)$ with M_c becomes its meridian, while $W^s(P_+)$ could become complicated, but its homology class does not contain the meridian component. This means that any point on $W^u(P_-) \cap S_c$ is accessible from the boundary ∂S_c.

Now we decrease c slightly below c_-. The Morse theory tells us how the level surface M_c changes in this process, and in particular how the intersection of $W^s(P_+)$ with M_c changes. At the moment when c reaches c_-, the level surface S_{c_-} is pinched at P_- and hence so is $W^s(P_+) \cap S_{c_-}$. At this moment, the pinched point P_- is still accessible from the boundary ∂S_{c_-}. Then, as c becomes slightly less than c_-, the surface M_c is cut at P_- and changes to a 2-sphere. Therefore the intersection of $W^s(P_+)$ on S_c is also cut at P_- and is now pinched at two points which correspond to $W^s(P_-)$. Even at this stage, the accessibility of the pinched points does not change, since local accessibility is preserved in this process.

Since the flow defined homeomorphism sends $W^s(P_\pm) \cap S_c$ with $c < c_-$ to C_{in} in D_{in}, this completes the proof of the accessibility of $W^s(P_-)$ in D_{in}. The same argument works for C_{out} in D_{out}. □

The accessibility of $W^s(P_-) \cap D_{in}$ and $W^u(P_+) \cap D_{out}$ should be preserved under topological equivalence of the flow, and hence the change of accessibility at least shows change of dynamical structure. As we already mentioned, when $\gamma = \gamma_0 = (1, 0, 0)$, the two-dimensional invariant manifolds $W^s(P_+)$ and $W^u(P_-)$ have a non-empty intersection along a unique heteroclinic orbit Γ_0 which, as we will show in Proposition 5.1, is transversal, C_{in} reduces to $(W^s(P_-) \cup W^s(P_+)) \cap D_{in}$ and $W^s(P_-) \cap D_{in}$ is accessible from ∂D_{in}. On the other hand, from the numerical results in [26] (see also [30]), it follows that there exists a value $\gamma_T = (\lambda_T, \mu_T, 0)$, with $\mu_T < 0$, at which $W^s(P_+)$ and $W^u(P_-)$ have an additional tangency along an orbit Γ_T and $W^s(P_-) \cap D_{in}$ contains points that are not accessible from ∂D_{in}. In Figure 4 we represent the intersection of the invariant manifolds of P_+ and P_- with the plane $x = 0$ as well as the related insets for $\gamma = \gamma_0 = \gamma_T$.

Let us provide a short description of the situation at $\gamma = \gamma_T$. One of the branches of $W^s(P_-)$, that we assume to cut D_{in} at s_1, has no intersection with $x = 0$ but the other one, cutting D_{in} at s_2, has two intersections with $x = 0$ at points A and A'. On the other hand, the intersection of $W^s(P_+)$ with $x = 0$ consists of three pieces Σ_i, with $i = 1, 2, 3$, determined by the points T and T', where T is a point where an orbit on $W^s(P_+)$ is tangent to $x = 0$ and T' is its image by the backward flow . Given points on a small fundamental domain of $W^s(P_+)$ and following their orbits backward in time, the first intersection with $x = 0$ occurs on Σ_1. Points on Σ_1 from T to Γ_0 have a second and a third intersection on Σ_2 and Σ_3, respectively. The behaviour of the intersection of the unstable manifolds with $x = 0$ is symmetric, with respect to $z = 0$, to that we have just described for the stable manifolds. Backward orbits of points on the curve Σ formed by the union of Σ_3 with the piece of Σ_1 from Γ_0 to T' will intersect D_{in} with no additional intersections with $x = 0$; note that backward orbits of points close to Γ_0 will cut D_{in} following the branch of $W^s(P_-)$ which intersects D_{in} at s_1. It is now clear that, when $\gamma = \gamma_T$, s_1 and the intersection with D_{in} of the piece of Σ from Γ_T to Γ_0 determine a closed curve containing $\{s_2\}$ in its interior, implying that s_2 is not accessible from ∂D_{in}.

It is clear that a thorough investigation of the accessibility of $W^s(P_-)\cap D_{in}$ would be very useful. This set is either one point or two points, and it in fact consists of two points when $\mu \geq 0$. On the other hand it is a single point if $\mu < 0$ and (λ, μ, ν) lies in a sufficiently small neighborhood of the saddle-node bifurcation set SN. Near the Bogdanov-Takens bifurcation point, this set changes from one point to two points at the homoclinic bifurcation set Hom. It seems difficult to study the similar transition in a neighborhood of the Hopf-Zero bifurcation point HZ, or along the μ-axis within the class of divergence-free (time-reversible) cases. Near these values, numerical methods, as those described in §4.3, can be helpful.

Figure 4: Non accessibility of $W^s(P_-) \cap D_{in}$ when $\gamma = \gamma_T$

4.2 Topological properties of the inset and outset

In this subsection, we collect some topological properties of the inset and outset, which are generally true for a broader class of systems than just (3.1).

When the inset and outset are non-empty and compact, we may speak about the continuity properties of the maps with respect to the Hausdorff topology. For a given parameter value γ, we consider the corresponding $C_{in}(\gamma)$ and $C_{out}(\gamma)$. When we consider the dependence of these sets with respect to the parameters, it is convenient to think of these sets also as set-valued maps $C_{in} = C_{in}(\gamma)$ and $C_{out} = C_{out}(\gamma)$. We abuse the notation C_{in} and C_{out} for denoting these maps as well.

Proposition 4.6 *The map C_{in} [resp. C_{out}] defined on a parameter region where it takes values in the set of non-empty compact subsets in D_{in} [resp. D_{out}], endowed with the Hausdorff topology, is upper-semicontinuous. Namely, for any sequence γ_n converging to γ_*,*

$$\limsup_{n\to\infty} C_{in}(\gamma_n) \subset C_{in}(\gamma_*)$$
$$\textit{[resp. } \limsup_{n\to\infty} C_{out}(\gamma_n) \subset C_{out}(\gamma_*)].$$

Recall that, for a family of sets $\{A_n\}$, we define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} A_k}.$$

Proof. Take any $p \in D_{in} \setminus C_{in}(\gamma_*)$. Because of the continuity of solutions of differential equations with respect to initial conditions and parameters, there exists a neighborhood U of p such that $U \subset D_{in} \setminus C_{in}(\gamma)$ for all γ close enough to γ_*. Since $\lim_{n\to\infty} \gamma_n = \gamma_*$, there exists some n_0 such that $U \subset D_{in} \setminus C_{in}(\gamma_n)$ for all $n \geq n_0$. Hence $p \in D_{in} \setminus \limsup_{n\to\infty} C_{in}(\gamma_n)$. Similar argument proves the statement for C_{out}. □

Remark 4.7 The maps C_{in} and C_{out} are in general not continuous as we saw in Theorem 4.2. Similar upper-semicontinuity property also holds for the maximal invariant set in V as well.

Proposition 4.8 *The inset and outset have isomorphic Alexander-Spanier cohomology groups:*

$$\bar{H}^k(C_{in}) \cong \bar{H}^k(C_{out}), \qquad (k \in \mathbb{Z}).$$

PROOF. From the definition of the traffic regulator, the transit map is a homeomorphism between the transit sets $D_{in} \setminus C_{in}$ and $D_{out} \setminus C_{out}$, and hence their corresponding (singular) homology groups are isomorphic. From the Alexander duality for a compact subset C in the interior of a closed 2-disk D^2 ([35]), we have

$$\tilde{H}_q(D^2 \setminus C) \cong \bar{H}^{1-q}(C), \qquad (q \in \mathbb{Z}),$$

and hence the conclusion follows. □

Remark 4.9 In general the Alexander duality does not hold if one replaces the Alexander-Spanier cohomology [35] by, say, the singular cohomology, unless the set C has a nice property, such as having the homotopy type of a CW complex. For instance if the space is connected but not arcwise connected, its singular cohomology and Alexander-Spanier cohomology may differ. For the examples of inset and outset that are discussed in this paper, it seems that the corresponding singular and Alexander-Spanier cohomologies always agree, and hence one might ask whether the same duality holds in our case for singular cohomology. However, we do not have any idea about how much this could be true.

Note that the cohomology groups $\bar{H}^k(C_{in})$ and $\bar{H}^k(C_{out})$ are non-trivial only when $k = 0, 1$ from the Alexander duality and from the fact that the (co)homologies of negative dimensions are trivial. One can also say a little more about cohomologies of the inset and outset. Firstly $\bar{H}^0(C)$ is, in general, a free Abelian group generated by the connected components of C. Similarly, the singular homology group $H_0(X)$ is a free Abelian group generated by its arcwise connected components of X, and in the case of $X = D^2 \setminus C_{in}$ or $D^2 \setminus C_{out}$, its connected components are arcwise connected, since it is an open set of D^2. Therefore its reduced singular homology $\tilde{H}_0(X)$ with $X = D^2 \setminus C_{in}$ or $D^2 \setminus C_{out}$, which is proven to be isomorphic to $\bar{H}^1(C_{in})$ or $\bar{H}^1(C_{out})$ respectively, is also determined by its (arcwise) connected components. In particular both of these cohomologies are free Abelian groups.

As we will see later, the inset and outset are not homeomorphic to each other in general. Again, it seems likely, at least in the examples that are given in this paper, that these two sets are homotopically equivalent.

The next proposition says that the cohomology of the inset and outset gives a lower estimate of the complexity of the cohomology of the invariant set $\mathrm{Inv}(V)$ itself.

Proposition 4.10 *There exist flow-defined maps*

$$\varphi^k_{in} : \bar{H}^k(C_{in}) \to \bar{H}^k(\mathrm{Inv}(V)),$$
$$\varphi^k_{out} : \bar{H}^k(C_{out}) \to \bar{H}^k(\mathrm{Inv}(V)),$$

for any $k \in \mathbb{Z}$, and these maps are injective.

Proof. First note that the second cohomology $\bar{H}^2(\mathrm{Inv}(V))$ is trivial, as it is isomorphic to $\tilde{H}_0(V \setminus \mathrm{Inv}(V))$ from the Alexander duality and $V \setminus \mathrm{Inv}(V)$ is (arcwise) connected. In fact if there were an extra component, any orbit starting a point in the component cannot leave the component at any time, and hence it must belong to $\mathrm{Inv}(V)$. Therefore the nontrivial maps are only for $k = 0, 1$ in the above. We shall prove the assertion for the inset, since the outset can be treated similarly.

As remarked before, the cohomology group $\bar{H}^0(A)$ is freely generated by the connected components of the set A. Suppose there are more than one connected components in C_{in}, then they are separated by two disjoint open sets in D_{in}, and therefore one can take a path whose end points are in ∂D_{in}, which does not intersect C_{in}, and which separates the two connected components of C_{in}. Since the path is entirely in the complement of C_{in}, any positive orbit starting at the path reaches the exit disk D_{out} in finite time. By definition, each connected component of C_{in} separated by the path must tend to a part of the invariant set inside V. Therefore the positive orbit of the path in V must separate two different connected component of the invariant set $\mathrm{Inv}(V)$. This means at least an injective map from connected components of C_{in} to those of $\mathrm{Inv}(V)$, which induces the map $\varphi^0_{in} : \bar{H}^0(C_{in}) \to \bar{H}^0(\mathrm{Inv}(V))$.

In order to prove the existence of φ^1_{in}, recall from the Alexander duality that $\bar{H}^1(C_{in}) \cong \tilde{H}_0(D_{in} \setminus C_{in})$ and that the reduced singular homology group $\tilde{H}_0(D_{in} \setminus C_{in})$ can be viewed as generated by the (arcwise) connected components of $D_{in} \setminus C_{in}$ which do not contain ∂D_{in}. Let X be such a component of $D_{in} \setminus C_{in}$ and choose a point x from it. Since x is in the complement of C_{in}, its positive orbit eventually reaches a point y in $D_{out} \setminus C_{out}$ and one obtains a path connecting those two points without intersecting $\mathrm{Inv}(V)$. Connecting x to a point in ∂D_{in} and y to the corresponding point in ∂D_{out} defines a loop in V not intersecting $\mathrm{Inv}(V)$ and not contractible in $V \setminus \mathrm{Inv}(V)$. Therefore it defines a non-trivial 1-cycle in $V \setminus \mathrm{Inv}(V)$, and hence a map from $\tilde{H}_0(D_{in} \setminus C_{in}) \cong \bar{H}^1(C_{in})$ to $\tilde{H}_1(V \setminus \mathrm{Inv}(V)) \cong \bar{H}^1(\mathrm{Inv}(V))$. Note that the last equality is again the Alexander duality applied to $\mathrm{Inv}(V) \subset V$. From the definition, this map is clearly injective. This completes the proof. $\square$

Finally we note that the inset and outset have some relation to the Conley index (see [29] for more information) of the maximal invariant set S inside the isolating neighborhood V. In fact V is nothing but a specific type of isolating blocks in the terminology of the Conley index theory. In particular, if the entrance domain is a single disk, then it is contractible and hence the Conley index of S is trivial. In other words, if the invariant set S has non-trivial Conley index, the entrance domain must be different from the single disk, say, more than one 2-disks or a domain which is not contractible. See [4] for some relevant cohomological properties of isolating blocks.

4.3 Numerical computation of the inset and outset

Results in §4.1 and §4.2 suggest that it may be helpful to consider more detailed information about the inset and outset. With that aim we propose a possible numerical method and illustrate a few resulting examples (see Figures 5 and 6). In Figure 5, we show how the inset changes when the parameters cross the homoclinic bifurcation set close to the Bogdanov-Takens bifurcation point. Figure 6 represents an approximation of the inset at the Kuramoto point which is discussed in the next section.

In all cases we consider the family (2.4) taking one of the parameters equal to -1 and the others in $\mathbb{R}^2$, as indicated in the figures. It is easy to check that, in all cases, when $x_0 < 0$ and $y_0 > 0$ the surface S given as the boundary of the domain $(-\infty, x_0] \times [y_0, \infty) \times (-\infty, 0]$ contains an entrance disk for a traffic regulator. Moreover, to simplify subsequent calculations we always assume that the vertex $(x_0, y_0, 0)$ of S is chosen to satisfy $\bar{\lambda} + \bar{\mu} y_0 + x_0^2 > 0$.

Let

$$\begin{aligned} R &= ((\{x_0\} \times [y_0, y_0 + r_y] \times [-r_z, 0]) \setminus P_\varepsilon) \\ &\quad \bigcup ([x_0 - r_x, x_0] \times \{y_0\} \times [-r_z, 0]) \\ &\quad \bigcup (([x_0 - r_x, x_0] \times [y_0, y_0 + r_y] \times \{0\}) \setminus Q_\varepsilon), \end{aligned}$$

Figure 5: Insets before (upper figure) and after (lower figure) the homoclinic bifurcation near the Bogdanov-Takens point.

Figure 6: Inset at Kuramoto point and its enlargement. Notice the fractal-like structure of the inset.

with

$$P_\varepsilon = \{x_0\} \times \left(-\frac{\bar{\lambda} + x_0^2}{\bar{\mu}} - \varepsilon, y_0 + r_y\right] \times (-\varepsilon, 0]$$

and

$$Q_\varepsilon = \left\{(x, y, z) \in [x_0 - r_x, x_0] \times [y_0, y_0 + r_y] \times \{0\} \mid y > -\frac{\bar{\lambda} + x^2}{\bar{\mu}} - \varepsilon\right\},$$

for some positive r_x, r_y, r_z and ε.

In our computational method for the insets, we start finding values of r_x, r_y, r_z and ε numerically such that any positive orbit of $p \in \partial R$ reaches a fundamental domain D^s of the stable manifold of the hyperbolic attractor at infinity. This fundamental domain is determined by the directional blowing-up in the positive y-direction. Notice that in such a case R is an entrance disk for a traffic regulator V, since all orbits cross R in the same direction because we have excluded P_ε and Q_ε.

A good choice of the vertex $(x_0, y_0, 0)$ and (r_x, r_y, r_z) makes V to contain the maximal invariant set. We then choose points from the entrance disk R and follow their positive orbits numerically. If a positive orbit reaches the fundamental domain D^s within some fixed time T which is indicated in each figure, then we color its initial point in such a way that the transit time is indicated by the darkness of the color. This gives the numerically obtained insets.

5 The divergence-free case

In this section we shall discuss the case when $\nu = 0$, namely the divergence of the vector field is equal to zero. Under this condition, the vector field (3.1) is equivariant under the transformation:

$$(x, y, z, t) \mapsto (-x, y, -z, -t),$$

and therefore is time-reversible.

5.1 On the transversality of heteroclinic orbits

When the system is divergence-free, it is known that the heteroclinic orbit proven in Theorem 3.1 is unique, see [27] and [38]. In these papers, the uniqueness is essentially based on the time-reversibility of (3.1) with $\nu = 0$, and hence one may not expect that the same holds in the general case where $\nu \neq 0$.

It was also pointed out by [30] that the uniqueness of the heteroclinic orbit in the above case implies the transversality of the two-dimensional stable and unstable manifolds along it, although the proof was not given there. For completeness let us provide a proof of it in a slightly more general situation. A heteroclinic orbit from P_- to P_+ is called *primary* if the x-component of the corresponding heteroclinic solution changes its sign only once, say at $t = 0$. Similarly, a heteroclinic orbit from P_- to P_+ is called *odd* if its x-component can be taken as an odd function of time t. Clearly, if the heteroclinic orbit in the time-reversible system with $\nu = 0$ is primary, it must be odd up to some time shift.

Proposition 5.1 *If there exists a primary heteroclinic orbit from P_- to P_+ in the system with $\nu = 0$ and $\mu \geq 0$, then along the heteroclinic orbit, $W^u(P_-)$ and $W^s(P_+)$ must intersect transversally.*

Note that here we do not need the uniqueness of the heteroclinic orbit. We need, however, that it is primary (and hence odd), as well as the specific form of our system, and therefore the proof cannot be easily generalized to other cases.

PROOF. Let $(x(t), y(t), z(t))$ be the corresponding heteroclinic solution with $x(t)$ being an odd function of t. We consider bounded solutions of the variational equation along the heteroclinic solution:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= w, \\ \dot{w} &= 2x(t)u + \mu v. \end{aligned} \tag{5.1}$$

Usual exponential dichotomy argument shows that any bounded solution of this equation must converge to zero exponentially as t tends to $\pm\infty$. There exists an obvious non-trivial bounded solution given by $(\dot{x}(t), \dot{y}(t), \dot{z}(t)) = (y(t), z(t), \lambda + \mu y(t) + x(t)^2)$. On the other hand, there must exist at least one unbounded solution, since the equilibrium points $P_\pm$ are of saddle-type. Therefore the space of all bounded solutions cannot be of dimension three, and hence the transversality of $W^u(P_-)$ and $W^s(P_+)$ is equivalent to the one-dimensionality of the set of all bounded solutions to (5.1). Suppose all bounded solutions form a two-dimensional space, and consider their initial conditions at $t = 0$. The non-trivial bounded solution $(\dot{x}(t), \dot{y}(t), \dot{z}(t))$ has the initial condition $(a, 0, c)$ with $a \neq 0$. Since the variational equation (5.1) is time-reversible, namely the equation does not change under the involution $(u, v, w, t) \mapsto (-u, v, -w, -t)$, the existence of a bounded solution with an initial condition (p, q, r) implies the existence of a bounded solution with the initial condition $(-p, q, -r)$. Since the dimension of the set of all bounded solutions cannot exceed two, these three vectors $(a, 0, c)$ and $(\pm p, q, \pm r)$ must be linearly dependent, and hence, either $q = 0$ or $a : c = p : r$. Therefore the only possibility for an extra bounded solution to exist is with the initial condition $(0, 1, 0)$ or $(0, 0, 1)$, without loss of generality.

Let $(u(t), v(t), w(t))$ be the solution of (5.1) with the initial condition $(0, 1, 0)$ and we shall derive a contradiction by using a shooting argument. For small enough $t > 0$, $u(t)$ and $w(t)$ are increasing and hence positive, as $\dot{u}(t) = v(t) > 0$ and $\dot{w}(t) = 2x(t)u(t) + \mu v(t) > 0$ for small enough $t > 0$. Note that here we have used $\mu > 0$ and $x(t) > 0$. This implies that $v(t)$ is also increasing initially as $\dot{v}(t) = w(t) > 0$. If this solution is converging to 0 as $t \to \infty$, $v(t)$ must turn to decreasing at some $t > 0$. Let $t_0 > 0$ be the first such t, namely, $\dot{v}(t_0) = 0$, $\dot{v}(t) \geq 0$ for all $t \in [0, t_0]$, but $\dot{v}(t) < 0$ for $t > t_0$ with t sufficiently close to t_0. Then there must exist some $t_1 \in (0, t_0)$ such that $\ddot{v}(t_1) < 0$. However $\dot{u}(t) = v(t) \geq 1$ for any $t \in [0, t_0]$ implies that $u(t) > 0$. This is a contradiction, since

$$0 > \ddot{v}(t_1) = \dot{w}(t_1) = 2x(t)u(t) + \mu v(t) > 0,$$

and therefore we have proven that the initial condition $(0, 1, 0)$ cannot give a bounded solution, hence $W^u(P_-)$ and $W^s(P_+)$ intersect transversally.

Similar argument yields contradiction as well in the case of the initial condition being $(0, 0, 1)$. This completes the proof of the assertion. $\square$

5.2 Bifurcations from a heteroclinic cycle

One interesting remark is that the system with $\nu = 0,\ \lambda \leq 0,\ \mu < 0$ has been studied extensively in connection with a partial differential equation called the Kuramoto-Sivashinsky equation ([25], [34]). Here we do not intend to discuss this equation, but only point out that our system (3.1) appears naturally as the traveling wave equation of this PDE, and many papers have been devoted to study the solutions of this equation. In particular, Kuramoto and Tsuzuki [25] found an explicit bounded solution of the equation when

$$\lambda = \lambda_K = -\frac{11 \cdot 15^2}{19^3}\eta^{\frac{3}{2}} \approx -0.310072,\ \mu = \mu_K = -\eta^{\frac{1}{2}} \approx -0.950713,\ \nu = 0,$$

where η is the unique real root of the cubic equation

$$\left(\frac{11 \cdot 15^2}{19^3}\right)^2 \eta^3 + \eta - 1 = 0.$$

This explicit solution corresponds to a heteroclinic orbit given as the intersection of one-dimensional invariant manifolds $W^u(P_+)$ and $W^s(P_-)$. Such a connection is far from being transverse, even for divergence-free vector fields, and hence will be destroyed by an arbitrarily small perturbation of the parameters λ and μ.

If, however, the two-dimensional invariant manifolds $W^u(P_-)$ and $W^s(P_+)$ have non-empty and transverse intersection at the moment when $W^u(P_+)$ and $W^s(P_-)$ intersect, then these intersections form a heteroclinic cycle connecting P_- and P_+. The existence of such a heteroclinic cycle implies interesting dynamical phenomena as we will show in the next theorem. Note that, although the transverse intersection of $W^u(P_-)$ and $W^s(P_+)$ is not proven rigorously, it seems very likely that it really exists according to numerical experiments of [26].

Theorem 5.2 *Suppose $W^u(P_+) \cap W^s(P_-) \neq \emptyset$ and $W^u(P_-) \cap W^s(P_+) \neq \emptyset$ at a parameter value $\gamma_* = (\lambda_*, \mu_*, 0)$. Suppose also that the intersection of $W^u(P_-)$ and $W^s(P_+)$ is transverse. Then the following bifurcations occur from the heteroclinic cycle formed by the intersections of these invariant manifolds of P_- and P_+:*

(1) *There exist two homoclinic bifurcation curves associated with P_- and P_+ which spiral in to γ_*. These two bifurcation curves are symmetric to each other with respect to the μ-axis, and hence they have infinitely many intersection points on the μ-axis at $\mu_i^\pm$, $\mu_i^- < \mu_* < \mu_j^+$, accumulating to μ_* from both sides of μ_* as i and j tend to ∞.*

(2) *Near each* $\mu_i^{\pm}$ *for large enough* i*, there exists at least one heteroclinic cycle of* P_- *and* P_+ *formed by a non-empty transverse intersection of* $W^u(P_-)$ *and* $W^s(P_+)$ *as well as a non-empty intersection of* $W^u(P_+)$ *and* $W^s(P_-)$*, called a subsidiary heteroclinic cycle.*

(3) *This subsidiary heteroclinic cycle has the same bifurcation structure in its unfolding as the primary heteroclinic cycle at* γ_**, and hence is accompanied by two symmetric homoclinic bifurcation curves which spiral in to the parameter point* $\gamma_i^{\pm}$ *corresponding to the subsidiary heteroclinic cycle, and each intersection points close enough to* $\gamma_i^{\pm}$*, there exist yet another subsidiary heteroclinic cycles having the same bifurcation structure in its neighborhood. Therefore the entire bifurcation set in a neighborhood of* γ_* *contains a self-similar structure. See Figure 7.*

Figure 7: Sketch of the bifurcation diagram for the codimension two heteroclinic cycle.

Numerical experiments carried out by Nishiyama [31] suggest that such a complicated bifurcation structure seems to occur not only at the Kuramoto point $\gamma_K = (\lambda_K, \mu_K, 0)$ but also at many other points on the μ-axis, and these points seem to converge to the Hopf-Zero bifurcation point $HZ = (0, -1, 0)$.

Note also that Kent and Elgin [21] have studied bifurcations from the same type of heteroclinic cycle as in the above theorem. They have discussed the similarity of the dynamics in a neighborhood of the cycle and that for the so-called Shil'nikov-type homoclinic orbit ([33, 20]). Glendinning and Sparrow [19] have also made a similar remark.

PROOF. First we note that the equilibrium points $P_\pm$ are hyperbolic and of saddle-focus type. The dynamics near a saddle-focus point with, say, the two-dimensional stable manifold and the one-dimensional unstable manifold is studied in the following way.

Take a small enough cylindrical neighborhood of the saddle-focus whose side boundary is transverse to the stable manifold and whose top and bottom boundaries are transverse to the unstable manifold. Then any segment in the side boundary, transverse to the stable manifold is mapped under the flow-defined local transition map near the saddle-focus point to a spiral curve in the top (or bottom) boundary.

Therefore, for the time-reversible heteroclinic cycle, $W^u(P_-)$ is accompanied by a part of $W^u(P_+)$ which comes from the one side of its transverse intersection to $W^s(P_-)$, and the intersection of this part of $W^u(P_+)$ at the (y, z)-plane forms a spiral curve whose center corresponds to $W^u(P_-)$.

Because of the time-reversibility, we have the same structure for $W^s(P_+)$ and $W^s(P_-)$ as well. In fact these two spirals are symmetric to each other with respect to the y-axis.

Since homoclinic bifurcation curves for, say, P_- are given by the intersection of $W^u(P_-)$ and $W^s(P_-)$, and since, generically, $W^u(P_-)$ moves in the (y, z)-plane diffeomorphically to the variation of parameters (μ, ν) in a neighborhood of γ_K, it is obvious that the homoclinic bifurcation curve for P_- appears as a spiral curve centered at γ_K. Similarly the homoclinic bifurcation curve for P_+ becomes a symmetric spiral with respect to the μ-axis. This proves the first assertion.

In order to find a subsidiary heteroclinic orbit, we shall look for intersections of the y-axis with the one-dimensional stable and unstable manifolds.

If we perturb the primary heteroclinic cycle within the class of time-reversible vector fields, namely perturb μ while keeping $\nu = 0$, then the spirals in the (y, z)-plane move symmetrically with respect to the y-axis. This means that the y-axis is mapped to a curve cutting through the spiral (without cutting the center of the spiral) in the top boundary of the cylindrical neighborhood of the saddle-focus, say, P_- under the time-reversed flow. Under the inverse of the local transition map, it is then mapped to a curve close to the stable manifold, if the perturbation is small enough.

On the other hand, $W^u(P_-)$ goes through a neighborhood of P_+ and may come back close to $W^u(P_+) \cap W^s(P_-)$, if the perturbation is chosen to be sufficiently small. In this case, a slight change of the parameter μ keeps major part of the image of the y-axis while the unstable manifold $W^u(P_-)$ cuts the image of the y-axis transversally. Therefore a subsidiary heteroclinic orbit must exists near a homoclinic bifurcation point. The second assertion is thus proved. The third assertion is merely a consequence of a similar reasoning applied to the subsidiary heteroclinic cycle. □

6 Concluding remarks

As readers should have noticed, we are still very far from complete understanding of the bifurcations in unfoldings of the triple zero singularity; there remain many questions that need to be answered. In what follows, we shall provide a list of open problems which we believe helpful for the future study of the bifurcations from the nilpotent singularity of codimension three.

6.1 Problems about elementary bifurcations

We believe that there are no additional bifurcations on and between the saddle-node bifurcation curve *SN* and the Hopf bifurcation curve H of singularities. Note that both of these curves are explicitly computed.

Numerical experiments ([31]) suggest that there exist a few curves of successive period doubling bifurcations for a periodic orbit which is born from the Hopf bifurcation. Such successive period doubling bifurcations may occur infinitely many times and accumulate to the onset of chaotic dynamics.

6.2 Problems related to the structure of the inset and outset

We believe the inset and outset for our systems always to be connected, although we have no proof of it. This is equivalent to saying that the maximal invariant set inside the box V is connected. In order to prove this, one may need to exclude the possibility of having bifurcations away from the previously existing dynamics. For instance, if there is a saddle-node bifurcation completely independent of the other dynamics, it would imply that the maximal invariant set is (or equivalently the inset and outset are) disconnected. We conjecture that this will not happen in the unfolding of the triple zero singularity.

If the inset and outset are connected, their 0-th cohomology groups $\bar{H}^0(C_{in})$ and $\bar{H}^0(C_{out})$ are isomorphic to $\mathbb{Z}$. Therefore one has only to look at the first cohomologies: $\bar{H}^1(C_{in})$ and $\bar{H}^1(C_{in})$. In §4 they are proven to be free Abelian groups generated by the (arcwise) connected components of the complements $D_{in} \setminus C_{in}$ and $D_{out} \setminus C_{out}$ which do not contain the boundaries ∂D_{in} and ∂D_{out}, respectively. These complements are certainly not connected in general, and hence the first interesting question in this direction would be whether these cohomologies are finitely generated, namely whether the complements have only finitely many components. We suspect that in general one cannot expect the finiteness of the number of components, for instance at the Kuramoto point $\gamma = \gamma_K$.

In order to study the structure of the inset and outset, the cohomology groups may be too rough. In §4, we saw that the inset and outset are in general not homeomorphic to each other, but they have isomorphic Alexander-Spanier cohomologies. So one may wonder whether these sets are homotopically equivalent in general. It is in fact the case for all the examples that are discussed in this paper, but we do not have any idea about how much this can be generalized. For sure there are examples of sets in a 2-disk which are not homotopically equivalent to each other but have homeomorphic complements; this is for instance the case for $C_1 = \{0\} \times [-1/2, 1/2]$ and

$$C_2 = C_1 \cup \{(x, \sin(1/x)/2) \mid 0 < x < 1/2\}.$$

Therefore answering this question may be difficult. A more interesting and certainly even more difficult question would be to classify the homotopy types of the inset and outset. We guess that if the complement of the inset has only finitely many components, then the inset is homotopically equivalent to a one-dimensional CW complex.

The main interest to study the inset and outset is the change of their structure as the parameters vary. As we have seen in Theorem 4.2, the inset becomes a 2-disk in a part of the parameter region called R_1. If we have a 2-disk as inset, its saturation by the flow gives a three-dimensional set which contains all the positive orbits that never reach the exit disk under the flow, and therefore we shall call it an *invariant pocket*. Theorem 4.2 shows that if $\gamma \in R_1$, then an invariant pocket exists, and moreover its boundary is the two-dimensional stable manifold of the equilibrium point P_-. One may be interested in how such an invariant pocket ceases to exist as the parameters are varied. Again from Theorem 4.2, if the parameters are varied in a small neighborhood of the Bogdanov-Takens point (BT), the invariant pocket is destroyed by the homoclinic bifurcation. We conjecture that this is in general the case. Along any choice of one-parameter subfamily which connects a point in R_1 and any other parameter point without an invariant pocket, the bifurcation where the invariant pocket ceases to exist is the homoclinic bifurcation of P_-.

As we have seen in the previous section, the inset and outset near the saddle-node bifurcation curve SN are completely described, and in particular they are contractible. Contractible inset and outset also appears in the divergence-free case when $\mu \geq 0$, namely when the system becomes gradient-like. According to the numerical calculation near the "cocoon" bifurcation discussed in [26], however, it seems likely that the inset and outset have a loop and hence are not contractible. It would be important to study how the structure changes between these two cases, along various paths in the parameter space. Also important is to study the structure of the inset and outset at other parameter values. In particular, we are interested in the structure of these sets near the Hopf-Zero point.

6.3 Near the Hopf-Zero point

The basic question about the bifurcation in our family near the Hopf-Zero point (HZ) is its genericity. Since the family (3.1) is a specific quadratic system, one must verify its genericity in order to apply the known results, such as those of Broer and Vegter [3]. One special property of our system is that it has a constant divergence ν and hence it is completely divergence free when $\nu = 0$. This already shows that the family (3.1) loses some kind of genericity with respect to the existence of invariant tori, see §2. However, this does not necessarily mean that it is not generic with respect to other bifurcations. We may expect that there exist parameter values, arbitrarily close to HZ, at which the system has a Shil'nikov type homoclinic orbit. For proving this, one may take different approaches: a standard way to prove it may be to use the Melnikov method. It is likely, however, that the Melnikov integral for the system (3.1) gives an exponentially small function of the parameters so that it is not easy to prove the existence of a homoclinic orbit in perturbations from HZ. Therefore the existence of the Shil'nikov homoclinic orbit in the system (3.1) still remains open.

It will also be helpful to study dynamics near HZ, if one knows the transversality of the two-dimensional stable and unstable manifolds of the equilibria $P_\pm$. It can be proven that these invariant manifolds have non-empty intersection at some parameter values, but it is not known whether they intersect transversally. Of course if the system can be made completely rotationally symmetric by a change of coordinates, the intersection cannot be transverse. For interesting information on this case we refer to [20] and especially to [24]. But the rotationally symmetric systems are not generic, and hence we can hope that the intersection likely becomes transverse. A proof of this has however not been given. One may again try to prove this by using a Melnikov-like method, but then one will face a similar difficulty of exponential smallness of the function.

6.4 The divergence free case

Along the curve DZ given by $\nu = 0$ in the parameter sphere, the corresponding systems are always divergence-free and at the same time time-reversible. When $\mu \geq 0$, the systems are also gradient-like and are known to have a unique heteroclinic orbit from P_- and P_+, from which the dynamics is completely described. When μ becomes negative, there are a number of bifurcations observed by numerical experiments, including the "cocoon" bifurcations [26], a codimension two heteroclinic orbit at the Kuramoto point, etc. Therefore the first problem is to understand such bifurcations in more detail. In particular, it will be interesting to know the first bifurcation that occurs when the parameter μ changes from positive to negative along DZ. Most likely the first bifurcation will be the heteroclinic bifurcation as discussed at the end of §4.1.

Again it will be helpful to have the transversality of the two-dimensional stable and unstable manifolds of $P_{\pm}$ along the μ-axis. The transversality is proven for $\mu \geq 0$ as discussed in §3, but the proof does not seem to work for negative μ. Numerical simulation suggests that these manifolds are always transverse.

A codimension two heteroclinic orbit was explicitly given at the Kuramoto point, but there are many more of such heteroclinic orbits at different parameter values, and it seems that parameter values for these codimension two heteroclinic orbits accumulate to HZ, according to numerical experiments. As discussed in §5, a cycle involving such a heteroclinic orbit implies subsidiary cycles and a self-similar bifurcation structure around it. Accumulation of such structures to HZ should be related to the existence of a Shil'nikov homoclinic orbit bifurcating from HZ.

6.5 More global bifurcations and computational approach

Away from the curve DZ, one loses the additional information that one has for zero-divergence, and hence studying dynamics and bifurcations will be much more difficult. One may be interested in tracing the parameter curve of homoclinic orbits emanating from BT. According to numerical experiments, it seems that the curve enters the parameter region where the Shil'nikov's eigenvalue condition is satisfied, inducing nearby chaotic dynamics due to the Shil'nikov's theorem; it then seems to approach the Kuramoto point on the curve DZ, namely a codimension two point of heteroclinic connection. From that point, other homoclinic orbits branch off and connect to other codimension two heteroclinic points, creating very complicated bifurcation structure.

There seems to be a rich variety of homoclinic bifurcations. Theoretical results for studying such homoclinic bifurcations would suggest to look for codimension two (or higher) homoclinic orbits. For instance a homoclinic orbit with the critical Shil'nikov condition quite likely exists, and will imply the existence of complicated dynamics including chaotic attractors, according to [32].

Recently new computational methods and tools have been developed, like HomCont/AUTO and GAIO, that could be used for studying the problems discussed in this paper.

HomCont/AUTO [6] can trace homoclinic and heteroclinic bifurcation curves in a parameter plane, and might hence be quite useful for studying homoclinic and heteroclinic bifurcations in the system studied in this paper. We can e.g. think at those curves emanating from BT or from HZ. Nishiyama [31] used an earlier version of HomCont and followed the homoclinic bifurcations curves from BT. He obtained results which seem to confirm the existence of a sequence of parameter points accumulating to HZ around each of which the bifurcations discussed in §5.2 occur. His study was rather incomplete and needs further attention. HomCont/AUTO might also be of great help to provide a computer assisted proof for the existence of such bifurcation points by using validated numerical computation.

Another interesting problem for which the numerical technique may be useful is to capture the two-dimensional invariant manifolds. This could lead to a numerical confirmation of the transversality of the two-dimensional stable and unstable manifolds discussed in §5.1. The numerical tool GAIO [16] is one of such tools, using the so-called subdivision algorithm for numerically approximating the two-dimensional objects in $\mathbb{R}^3$; see also [23] for a different approach. These tools may hence be of great help in tackling the problems discussed in §6.4.

A Proof of Theorem 4.2

Here we give a proof of Theorem 4.2. First we shall state a lemma which will be used later.

Lemma A.1 *Let X be a compact connected metric space consisting of more than one point and $H_0 \subset X$ be homeomorphic to a closed segment with end points x_1 and x_2. Assume that there exists a family of nonempty sets $H_\gamma \subset X$, with $\gamma \in B(\gamma_0, r)$ for some $\gamma_0 \in \mathbb{R}^n$ and $r > 0$, such that the following properties hold:*

(1) *There exist $x_1(\gamma)$, $x_2(\gamma) \in H_\gamma$ such that $x_1(\gamma) \to x_1$ and $x_2(\gamma) \to x_2$ when $\gamma \to \gamma_0$.*

(2) *For each* $r > 0$ *there exists* $s(r) > 0$ *such that if* $\gamma \in B(\gamma_0, s(r))$ *then* $H_\gamma \subset \{x \in X \mid d(x, H_0) < r\}$.

Then $H_\gamma \to H_0$ *in the Hausdorff metric as* $\gamma \to \gamma_0$.

The proof of this lemma is quite simple and hence omitted.

Proof of Theorem 4.2. Our knowledge of the local bifurcations arising along the line $SN = \{\gamma \in S^2 \mid \lambda = 0\}$ when $\mu \neq -1$, given in Theorem 2.5, is the main tool to prove the result. We will consider the versal unfoldings of the singularities (see [9]) to determine the dynamics inside the box V.

If $\gamma \in SN$ with $\mu < 0$ and $\nu < 0$, then the local family is a generic unfolding of a saddle node bifurcation with a (C^0, C^∞) versal unfolding given by the one-parameter family

$$\begin{aligned} x' &= \alpha + x^2 \\ y' &= -y \\ z' &= -z, \end{aligned} \tag{A.1}$$

where α is a real parameter. If $\gamma \in SN$ with $\mu > 0$ and $\nu \leq 0$, then a (C^0, C^∞) versal unfolding is given by the one-parameter family

$$\begin{aligned} x' &= \alpha + x^2 \\ y' &= -y \\ z' &= z, \end{aligned} \tag{A.2}$$

where α is a real parameter. Finally, close to BT, the local family is a generic unfolding of the Bogdanov-Takens bifurcation whose (C^0, C^0) versal unfolding can be given as

$$\begin{aligned} x' &= y \\ y' &= \alpha + \beta y + x^2 - xy \\ z' &= -z, \end{aligned} \tag{A.3}$$

where α and β are real parameters. In this last case it is useful to recall the bifurcation diagram of the singularity (see Figure 8). We do not need to know the local bifurcations at HZ; we only use the fact that, as mentioned in §2, the vector field is topologically equivalent to a vertical one at HZ.

Figure 8: Bogdanov-Takens bifurcation. In our three-dimensional family, there exists a two-dimensional center manifold on which such bifurcations occur. The center manifold is always normally attracting.

Taking into account the versal unfoldings (A.1), (A.2) and (A.3) and the local behavior at HZ, easily follows the existence of a small box V_γ, satisfying similar properties to those of V, where one can describe all the possible local bifurcations. To be precise, the boundary of V_γ consists of $D_{\gamma,in}$, $D_{\gamma,out}$ and K_γ. $D_{\gamma,in}$ and $D_{\gamma,out}$ are 2-disks where the vector field is transverse inward on $D_{\gamma,in}$, outward on $D_{\gamma,out}$. K_γ is homeomorphic to a cylinder $S^1 \times [0,1]$ and it is a flow box. On $D_{\gamma,in}$ and $D_{\gamma,out}$ we can again define the notion of inset and outset but with respect to V_γ. In order to avoid complicated notation we will refer to V_γ as the local box, and to $D_{\gamma,in}$ and $D_{\gamma,out}$ as the local entrance disk and the local exit disk, respectively. Similarly, we will also use the terminology local inset and local outset.

When $\lambda = 0$, and therefore $\alpha = \beta = 0$ in (A.1), (A.2) and (A.3), the change of the local inset and outset when γ varies can be described as follows:

(1) At HZ the local inset and outset reduce to a unique point.

(2) If $\gamma \in SN$ with $\mu < 0$ and $\nu < 0$, then the local inset is a 2-disk whose boundary is given by the intersection of the two-dimensional stable manifold of the singularity with the local entrance disk. The local outset reduces to a unique point.

(3) If $\gamma \in SN$ with $\mu > 0$, then the local inset and the local outset are homeomorphic to closed segments. Their end points are given by the intersection of the stable manifold and the unstable manifold at the singularity with the local entrance disk and the local exit disk, respectively.

(4) At BT, the local outset is a point, and the local inset is homeomorphic to a segment with end points given by the intersection of the stable manifold at the singularity with the local entrance disk.

Since $L(x, y, z) = z - \mu x - \nu y$ is a Lyapunov function when $\lambda = 0$, it follows that positive and negative orbits stay inside V only if they have the singularity as ω- and α-limit set, respectively, and also that there is no non-trivial orbit Γ such that $\alpha(\Gamma) = \omega(\Gamma) = O$. Therefore, taking V_γ smaller if necessary, we can assume that all the orbits which leave the local box V_γ also leave V; note that in such a case we get the description of the inset and outset for V along $\{\gamma \in S^2 \mid \lambda = 0\}$ since they are homeomorphic to the local ones. On the other hand, because of the continuity of the solutions with respect to the parameters, such property also holds for parameters in a neighborhood of a fixed γ with respect to the box V_γ. Since the dynamics inside such a box is given by the respective versal unfoldings, we only need to pay attention to possible bifurcations in order to complete the description of inset and outset of V close to the saddle-node points and the Bogdanov-Takens point. In the sequel we only pay attention to the local inset and outset.

After the saddle node bifurcation we find two singularities inside V_γ. If such bifurcation occurs for μ negative then one of the singularities is a hyperbolic attractor while the other one is a hyperbolic saddle with a two-dimensional invariant manifold. Hence the local inset is again a 2-disk whose boundary is given by the intersection of such two-dimensional stable manifold with the local entrance disk. The attractor belongs to the region enclosed by the two-dimensional stable manifold and the local entrance disk. Therefore, the local outset reduces to a unique point. If the saddle node bifurcation occurs for μ positive, then both singularities are hyperbolic saddles with different stability index. The local inset and outset are homeomorphic to segments whose end points are given by the intersection of the one-dimensional invariant manifolds of the singularity with the correspondent local entrance and exit disks.

At BT we must pay attention to the complete bifurcation diagram (see Figure 8). For values of the parameter close enough to BT and with $\lambda < 0$, two singularities s and e appear (see Figure 9). Let U be a neighborhood of BT where the local bifurcations are those given by the versal unfolding. The curve *Hom* of homoclinic bifurcation at s divides U into two regions R_1 and R_2 (see Figure 3). For parameters in R_1 the local inset is again homeomorphic to a 2-disk whose boundary is limited by the two-dimensional stable manifold of s and the local outset reduces to a unique point. The Hopf bifurcation at e has no influence on the shape of the local inset, and only the attractor inside the local box changes from e to a periodic orbit. For parameters in R_2 the local inset and the local outset are homeomorphic to a segment. The end points are given by the intersection of the stable manifold of e and the unstable manifold of s, respectively, with the local entrance disk. Therefore along *Hom* the qualitative change of the inset and the outset is understood. In Figure 9, we depict the local phase portraits for the vector fields in normalized coordinates. We see that for parameters at *Hom* the local inset is given by the union of two 2-disks connected by a segment. The union of the boundary of the disks and the segment is given again by the intersection of the stable manifold of s with the local entrance disk.

It remains to prove that insets and outsets change continuously with respect to the parameters outside the line *Hom*. Because of Proposition 4.6, we only need to prove the lower-semicontinuity. If for some value of γ the inset or the outset reduces to a unique point it is easy to check, by taking into account the continuous dependence of the solutions with respect to the parameters, that for any arbitrarily small neighborhood B of such a point and for values of the parameters close enough to γ, the corresponding inset or outset is contained in B. When the inset is a closed disk, we only need to prove that its boundary changes continuously. This fact follows from the

Figure 9: Qualitative change of the inset near the line of homoclinic bifurcation.

continuous dependence of the stable manifold with respect to the parameters when we take γ with $\lambda \neq 0$. If we consider γ with $\lambda = 0$ we must also take into account that (A.1) is a (C^0, C^∞) versal unfolding. Similar arguments are valid when the inset or the outset are homeomorphic to a segment and the parameter is not BT.

The proof of the continuity at BT is slightly more involved and relies on Lemma A.1. To apply it we consider $\gamma_0 = BT$ and H_0 as the inset for BT. We only need to check that the family of insets for parameters close to BT satisfies the property (1) stated in Lemma A.1. Note that x_1 and x_2 are the intersections of the stable manifold at the origin with the entrance disk when $\gamma = BT$. We distinguish three different cases:

(1) If $\gamma \in \{\gamma \mid \lambda = 0\}$ with $\mu > 0$, then we take $x_1(\gamma)$ and $x_2(\gamma)$ as the intersections of the stable manifold at the origin with the entrance disk.

(2) If $\gamma \in \{\gamma \mid \lambda = 0\}$ with $\mu < 0$, then we take $x_1(\gamma)$ and $x_2(\gamma)$ as the intersections of the strong stable manifold at the origin with the entrance disk.

(3) If we consider γ with $\lambda < 0$, then we take $x_1(\gamma)$ and $x_2(\gamma)$ as the intersections of the strong stable manifold at s with the entrance disk.

Since (A.3) is a (C^0, C^0) versal unfolding, it follows that the invariant manifolds used to define $x_1(\gamma)$ and $x_2(\gamma)$ converge to the stable manifold at the origin for BT when the parameter tends to BT. Hence Property (1) follows.

At *Hom* the continuity of the outset is straightforward. Let us now show that the inset function $C_{in}(\gamma)$ is not lower-semicontinuous. In fact we can reduce the proof to the case of local insets and therefore consider normalized coordinates as those depicted in Figure 9. Take $\gamma_0 \in \mathit{Hom} \setminus BT$ and let F_0 denote the focus point of the vector field X_{γ_0}. It suffices to look at a small transverse cylinder C around the stable manifold at F_0. Let A be one of the circles given by the intersection of C with the local entrance disk at γ_0. It is then possible to find a sequence $\{\gamma_n\}$ converging to γ_0 such that the corresponding $\limsup_{n\to\infty} \widehat{C}_{in}(\gamma_n)$, where $\widehat{C}_{in}$ denotes the local inset, cuts the circle A at any given point. See e.g. [12] for more details. This completes the proof since we already know that the local inset at $\gamma = \gamma_0 \in \mathit{Hom} \setminus BT$ is homeomorphic to the two 2-disks connected by a segment which entirely contains the circle A. $\square$

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